BSM in the sky

Tutorial

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The first three problems will be treated during the first tutorial session and the last three problems during the second tutorial session. We actively encourage you to work on the problems during the free time from 1 pm to 3 pm after lunch. Then, we kindly invite you to volunteer to present your solution on the black board. In any case, you will receive help from the tutor.

Some numerical values: 1 s $\simeq 1.5\times10^{24}$ GeV^{-1}, 1 cm $\simeq 5.1\times10^{13}$ GeV^{-1}, $M_{\rm pl} \simeq 2.44\times10^{18}$ GeV.

Problem 1 - Survival of the baryon abundance

We assume that we live in a universe with as many baryons as antibaryons and we estimate the proton-antiproton annihilation cross-section to be

$$\sigma_{p\bar{p}}\, v_{\rm rel} = \frac{c_1}{m_\pi^2}, \tag{1}$$

where c_1 is an order 1 constant.

- **Question 1 :** Using the instantaneous freeze-out approximation, compute the surviving relic abundance of nucleons and anti-nucleons.
- **Question 2 :** The baryon-to-photon ratio is infered by BBN prediction vs observation [1] to be $\eta \equiv n_B/n_\gamma \simeq 6.2 \pm 0.4 \times 10^{-10}$. Deduce the value of the baryon-to-entropy ratio n_B/s. We assume the effective number of neutrinos to be the SM value, namely $N_{\rm eff} \simeq 3.045$ [2, 3]. Conclude.

Problem 2 - Upper bound on the mass of thermal DM

The unitarity of the S-matrix $S^\dagger S$ is just the matrix form of the conservation of the occupation probability of a quantum state $\sum_f P_{i\to f}$. We can split the channels into elastic and inelastic scatterings $P_{\rm elast} + P_{\rm inelast} = 1$. Thanks to the orthonormality properties of the spherical harmonics, the conservation of the probability is even satisfied for each partial wave L, $P^{(L)}_{\rm elast} + P^{(L)}_{\rm inelast} = 1$.[1] Next, the inelastic cross-section of a 2-to-2 process can be decomposed into partial waves with each coefficient expressed as a function of $P^{(L)}_{\rm inelast}$

$$\sigma_{\rm ine} = \sum \sigma^{(L)}_{\rm ine} \quad \text{with} \quad \sigma^{(L)}_{\rm ine} = \frac{\pi(2L+1)}{p_i^2} P^{(L)}_{\rm inelast}. \tag{2}$$

[1] Partial waves are eigenstates of the square of the angular momentum $\mathbf{L}^2$.

This is how, in 1990, Griest and Kamionskowski (have a look at [4] but also app. B if you are interested) convert the unitarity requirement on an upper-bound on the annihilation cross-section

$$P_{\rm inelast}^{(L)} \lesssim 1 \qquad \rightarrow \qquad \sigma_{\rm ine}^{(L)} \lesssim \frac{\pi(2L+1)}{p_i^2}. \tag{3}$$

- **Question 1 :** Derive the non-relativistic limit of eq. (3) as a function of the relative velocity $v_{\rm rel}$ and the DM mass $M_{\rm DM}$.
- **Question 2 :** Deduce the upper bound on the mass of Majorana (self-conjugate) and Dirac (non self-conjugate) thermal DM assuming that it freezes-out by annihilating through partial wave L. You can use the value in [5] for the annihilation cross-section of thermal DM at freeze-out $\langle \sigma v_{\rm rel} \rangle_{\rm FO}$. We give $\langle 1/v_{\rm rel} \rangle = \sqrt{x_{\rm FO}/\pi}$ [4].
- **Question 3 :** However, for such heavy DM, we expect the coupling constant to be large and non-perturbative effects, the so-called Sommerfeld effects, to modify the velocity dependence of the annihilation cross-section (you can have a look at app. (A) for a quick summary) such that the value of $\langle \sigma v_{\rm rel} \rangle_{\rm FO}$ from [5] must be corrected. Understand why. Estimate how the upper bound on the DM mass from the previous question is affected.
- **Question 4 :** Show that a cross-section larger than few times the geometrical cross-section would violate unitarity.

Problem 3 - Relaxing the unitarity bound by injecting entropy

Consider the presence of an heavy cold relic V of mass m_V which dominates the energy density of the universe and decays into SM radiation after the DM has frozen-out. During the decay, non-relativistic degrees of freedom held in the cold relic are converted into relativistic degrees of freedom held in the radiation, hence creating entropy. We assume that the cold relic has already decoupled from the SM (and from DM) when it dominates the energy density of the universe, meaning that its comoving number density Y_V is conserved. We assume that the decay occurs instantaneously when $H \sim \Gamma_V$.

- **Question 1 :** Compute the temperature $T_{\rm dom}$ at which the cold relic starts dominating the energy density of the universe as a function of Y_V and m_V.
- **Question 2 :** Compute the temperature of the universe just after the decay $T_{\rm dec}^{\rm after}$.
- **Question 3 :** Compute the temperature of the universe just before the decay $T_{\rm dec}^{\rm before}$.
- **Question 4 :** Compute the dilution factor $D \equiv S_{\rm dec}^{\rm after}/S_{\rm dec}^{\rm before}$ as a function of m_V, Γ_V and Y_V, where S is the total entropy $S = s\, a^3$ and s is the comoving entropy. We can neglect the entropy stored in the cold relic before it decays. Realize how simple is the expression for D when expressed as a function of $T_{\rm dom}$ and $T_{\rm dec}^{\rm after}$.
- **Question 5 :** Compute the impact of the entropy injection due to the decay of the cold relic on the annihilation cross-section at freeze-out. Deduce the consequence for indirect detection.
- **Question 6 :** Compute the new unitarity bound on the mass of thermal DM as a function of the dilution factor D.
- **Question 7 :** The good agreement between theory and observation regarding Big-Bang Nucleosynthesis constrains the lifetime of any heavy cold relic to be smaller than ~ 0.03 s. Assuming $m_V = 100$ PeV, compute the maximal dilution factor and the maximal unitarity bound on the DM mass, compatible with BBN.

Problem 4 - Gamma-ray at Earth from DM decay

- **Question 1 :** Give a list of dimension 5 operators inducing the decay of DM, assuming that it is a singlet under the symmetry groups of the SM $G_{\rm SM}$. At least 3 operators for a singlet scalar DM and 1 operator for a singlet fermion DM would be nice.
- **Question 2 :** Deduce an estimate for the DM lifetime τ as a function of the cut-off Λ.
- **Question 3 :** Knowing that lower bound on the DM lifetime coming from gamma-ray and X-ray telescopes is roughly $\tau \sim 10^{26}$ s [6, 7] and assuming $\Lambda \sim GUT$, deduce an upper bound on the mass of DM decaying via a dimension 5 operator.

Problem 5 - Gamma-ray at Earth from DM annihilation

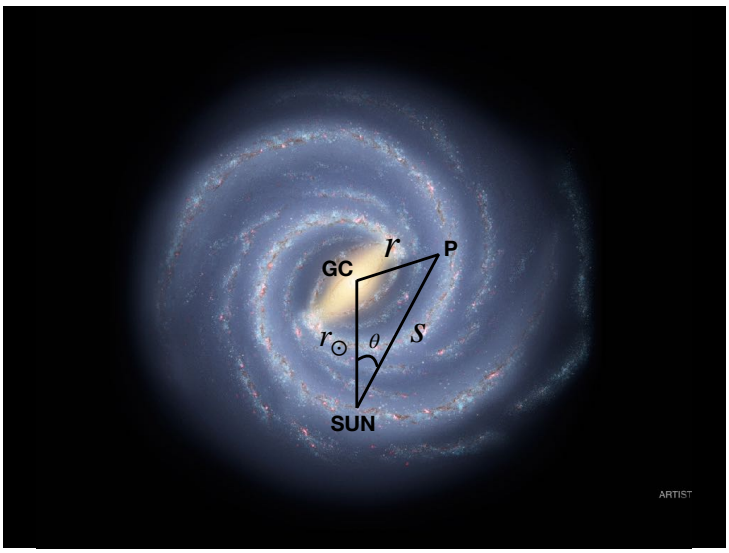


Figure 1: Relation between the angle θ and the distance along the line-of-sight s in eq. (7).

The detected gamma flux per solid angle unit and per energy unit from annihilation in the Milky Way of self-conjugate DM is (see app. C for a derivation)

$$\frac{d\phi_\gamma}{d\Omega dE} = \frac{1}{2}\frac{r_\odot}{4\pi}\left(\frac{\rho_\odot}{M_{\rm DM}}\right)^2 J(\theta)\sum_f \langle\sigma v\rangle_f \frac{dN_{f\to\gamma}}{dE} \tag{4}$$

where

$$J(\theta) = \int \frac{ds}{r_\odot}\left(\frac{\rho_{DM}(\vec{r})}{\rho_\odot}\right)^2 \tag{5}$$

is the so-called J factor for annihilation, $r_\odot = 8.33$ kpc is the Earth distance to the GC and $\rho_\odot = 0.3$ GeV/cm^3 is the supposed DM energy density at the Earth position. It is also common to define the J-factor averaged over a disk $[0,\ \theta]$ or an annulus $[\theta_1,\ \theta_2]$

$$\bar{J} = \frac{1}{\Delta\Omega}\ 2\pi \int_{\theta_1}^{\theta_2} d\theta \sin\theta\ J(\theta), \tag{6}$$

where $\Delta\Omega = 2\pi \int_{\theta_1}^{\theta_2} d\theta \sin\theta$. As shown in Fig.(1), The distance s between the Earth and the observation point P is related to the distance r between P and the galactic center GC through

$$r = \sqrt{r_\odot^2 + s^2 - 2r_\odot s\cos\theta} \tag{7}$$

where θ is the angle between the observer line-of-sight and the GC. A mostly used DM profile is the so-called Navarro-Frenk-White [8] fitted on N-body simulations

$$\rho_{\rm NFW}(r) = \rho_s \, \frac{r_s}{r} \frac{1}{\left(1+\frac{r}{r_s}\right)^2} \tag{8}$$

We choose the parameter $\rho_s = 24.42$ GeV/cm^3 and $r_s = 0.184$ kpc in order for the local DM density at the sun position $r_\odot = 8.33$ kpc to be $\rho_\odot = 0.3$ GeV/cm^3 and for the total DM mass contained within 60 kpc to be $M_{60} \equiv 4.7 \times 10^{11}\ M_\odot$ [9]. However, the presence of a cusp (DM profile peaked as $\rho_{DM} \sim 1/r$ at the center) contradicts the observation of a core in dwarf galaxies ($\rho_{DM} \sim r^0$ at the center). More precisely, it has been found that the star rotation velocity in the inner part of the galaxy shows a solid-body behaviour (rises linearly with the radius) hence indicating the presence of a central core in the DM distribution [10–17]. In order to agree better with the observation, an alternative to NFW is the iso-thermal profile (e.g. [18])

$$\rho_{\rm iso}(r) = \frac{\rho_s}{1+\left(\frac{r}{r_s}\right)^2}. \tag{9}$$

An other alternative (which is the one we choose here) is to modify NFW by including a core of size r_0

$$\rho^{\rm core}_{\rm NFW}(r) = \rho_s \, \frac{r_s}{r+r_0} \frac{1}{\left(1+\frac{r}{r_s}\right)^2}. \tag{10}$$

- **Question 1 :** Compute numerically the J-factor averaged over the disk $[0,\ 1°]$ centred on the GC, assuming the NFW profile in eq. (8).
- **Question 2 :** Repeat the exercise with the modified NFW profile in (10) including a core of size $r_0 = 0.5$ kpc. Understand how uncertainties on the presence of a core in the MW center translate on uncertainties on the DM annihilation cross-section upper bounds coming from indirect detection experiments. Understand also how these uncertainties vary when we vary the observation angle θ.
- **Question 3 :** The HESS (TeV gamma-rays telescope) collaboration provides one of the most stringent indirect detection constraints on annihilation cross-section of DM with a TeV mass. In [19], they use a NFW profile (without a core) averaged over the annulus $[0.3°,\ 1°]$. Estimate the potential error on the upper bound on the DM annihilation cross-section if the MW has indeed a ~ 0.5 kpc core.
- **Question 4 :** If you can not use numerical methods, you can try to prove analytically that the J-factor for the only-solid-core DM profile

  $$\rho_{\rm disk} = \rho_0 \qquad \text{if } r < r_0 \text{ and 0 otherwise} \tag{11}$$

  is $J_{\rm disk} = \frac{4\pi}{3}\left(\frac{\rho_0}{\rho_\odot}\right)^2\left(\frac{r_0}{r_\odot}\right)^3$. An advice, you should better rename $r_\odot \to d$ to prevent mixing up r_0 and $r_\odot$.

Problem 6 - Analogue of Sommerferld enhancement in classical gravity

Consider a point particle impinging on a star of radius R. Neglecting gravity, only particles with impact parameter b smaller than R will hit the star. Hence, the cross-section for the particle to crash on the star is just the geometrical cross-section $\sigma_0 = \pi R^2$. However, after including the

long range Newton's force, particles can crash on the star for larger impact parameter. Then, the cross-section is $\pi b_{\rm max}^2$, where $b_{\rm max}$ is the largest impact parameter leading to a crash. We denote v the velocity at infinity.

- **Question :** Show that
$$\sigma = \sigma_0 \left(1 + \frac{v_{\rm esc}^2}{v^2}\right), \tag{12}$$
where $v_{\rm esc}^2 = 2G_{\rm N}M/R$ is the escape velocity from the surface of the star. We conclude that a long-range Newton's interaction enhances the cross-section σv by of factor growing as $1/v$. This is a classical counterpart of the Sommerfeld enhancement, c.f. app. A.

Problem 7 - γ-ray constraints on $U(1)_D$ model

We have seen in problem 1 that thermal DM must be protected against decay via dimension 5 operators. An easy solution is to charge the DM (Dirac fermion X) under a local $U(1)_D$, hence also providing an interaction between DM particles mediated by a dark photon V_μ

$$\mathcal{L} \supset -\frac{1}{4}F_{D\mu\nu}F_D^{\mu\nu} - \frac{1}{2}m_V^2 V_\mu V^\mu + \bar{X}\left(i\not{D} - M_{\rm DM}\right)X \tag{13}$$

where $M_{\rm DM} > m_V$ are the masses of the fermion DM and vector mediator and $D_\mu = \partial_\mu + ig_D V_\mu$ is the covariant derivative. We define the dark fine structure constant $\alpha_D \equiv g_D^2/(4\pi)$. The dark sector can communicate to the SM via a renormalizable kinetic mixing

$$\mathcal{L} \supset -\frac{\epsilon}{2c_w}F_{D\mu\nu}F_Y^{\mu\nu}. \tag{14}$$

This allows the dark photon to decay into SM particles. We suppose that the dark sector and the SM have been at thermal equilibrium in the early universe (thermal DM scenario) but have decoupled well before DM freezes-out. DM freezes-out by annihilating into a pair of dark photons with cross-section

$$\sigma v_{\rm rel} = \frac{\pi\alpha_D^2}{M_{\rm DM}^2}. \tag{15}$$

We know fix the DM mass $M_{\rm DM} = 1$ TeV.

- **Question 1 :** Assuming that the perturbative cross-section σv in eq. (15) is correct, give the value of the dark fine structure constant α_D needed to correctly reproduce the correct DM abundance. We give $\langle\sigma v_{\rm rel}\rangle_{\rm FO} = 2.4 \times 10^{-26}$ cm^3/s.[2].
- **Question 2 :** We want to use indirect detection to constrain thermal DM with TeV mass range. A good experiment is HESS telescope which measures TeV energy γ-ray. Assuming the validity of the perturbative cross-section in eq. (15), discuss if the HESS constraints from [20] can probe the $U(1)_D$ model.
- **Question 3 :** For heavy thermal DM ($\sim$TeV mass range), the coupling constant α_D is large and non-perturbative effects called Sommerfeld effects can enhance the perturbative cross-section (see app. A for a bit of more details). The relative velocity in the MW which is $v_{\rm rel} \sim 220$ km/s $\sim 10^{-3}$ has to be compared to the relative velocity at freeze-out ~ 0.3. Compute the ratio of the annihilation cross-section in MW to the one at freeze-out $\langle\sigma v\rangle|_{\rm MW}/\langle\sigma v\rangle|_{\rm FO}$, assuming the Coulomb approximation (c.f. eq. (23)) to be valid.[3]

[2]The similarity of $\langle\sigma v_{\rm rel}\rangle_{\rm FO}$ with the s-wave perturbative Majorana value computed in [5] comes from the almost exact cancellation between the Dirac-to-Majorana factor and the velocity dependence of the non-perturbative (Sommerfeld-enhanced) cross-section $\langle\sigma v_{\rm rel}\rangle \propto 1/v_{\rm rel}$.

[3]Don't lose your time performing the average over the velocity distribution of DM in the galaxy and assume $\langle\sigma v_{\rm rel}\rangle = \sigma v_{\rm rel}\big|_{v_{\rm rel}=\sqrt{\langle v_{\rm rel}^2\rangle}}$.

Deduce the relevance of the Sommerfeld enhancement for indirect-detection in the MW. And in Dwarfs ?

- **Question 4 :** Using the expression for the Sommerfeld enhancement with finite mediator mass in eq. (25), compute the upper bound on the mediator mass m_V of the $U(1)_D$ model coming from HESS [20] assuming the branching ratio $BR(V \to \tau^-\tau^+) \sim 15$ % and the DM mass $M_{\rm DM} = 1$ TeV.

A What is the Sommerfeld enhancement ?

The Sommerfeld enhancement is a non-relativistic effect arising in Quantum Mechanics. Suppose that a non-relativistic particle is moving along the z direction with wavefunction

$$\psi_k^{(0)}(\vec{r}) = e^{ikz}, \tag{16}$$

and can be annihilated and converted into an other state due to a short-range interaction at the origin: $H_{\rm ann} = U_{\rm ann}\delta(\vec{r})$ [21]. Then the rate of the process will be proportional to $|\psi_k^{(0)}(\vec{r})|^2 = 1$. But now if we add a long-range interaction $V(\vec{r})$, the wavefunction of the particle will be distorted and this will change the annihilation rate. More precisely, the wavefunction $\psi_k(\vec{r})$ will obey to the Schrodinger equation

$$\left[-\frac{1}{2M}\nabla^2 + V(\vec{r}) + U_{\rm ann}\delta(\vec{r})\right]\psi_k = \epsilon_k\,\psi_k. \tag{17}$$

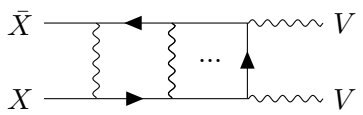
Since the annihilation takes only place locally in $r = 0$, the only effect of the long-range force is to modify the value of the wavefunction at the origin. This changes the cross-section

$$\sigma = \sigma_0 S_k, \tag{18}$$

by a factor called the Sommerfeld factor

$$S_k = \frac{|\psi_k(0)|^2}{|\psi_k^{(0)}(0)|^2}. \tag{19}$$

σ_0 is the perturbative cross-section, computed when taking only into account the short-range potential. From the point of view of Feynman diagrams, the Sommerfeld enhancement can be visualized as an infinite ladder of mediator exchange.



(20)

The infinite number of mediator exchange manifests the existence of bound states in the theory when the range of the Dark force $1/m_V$ is larger than the size of the would-be bound state system $(\mu v_{\rm rel})^{-1}$, μ being the reduced mass $M_{\rm DM}/2$ of the 2-body system

$$\text{BSE:} \qquad \frac{\alpha_D M_{\rm DM}}{2m_V} \geq 1. \tag{21}$$

We can neglect the mass of the mediator when the range of the Dark force $1/m_V$ is larger than the De Brogglie wavelength $(\mu v_{\rm rel})^{-1}$, the so-called Coulomb approximation

$$\text{Coulomb:} \qquad \frac{M_{\rm DM} v_{\rm rel}}{2m_V} \geq 1. \tag{22}$$

In that case, the Sommerfeld enhancement factor of the cross-section is simply

$$S_C = 2\pi \frac{\alpha_D}{v_{\rm rel}} \frac{1}{1 - e^{-2\pi \frac{\alpha_D}{v_{\rm rel}}}}. \tag{23}$$

In the opposit case, in presence of a finite mediator mass there is no exact analytical expression of the Sommerfeld enhancement factor. However, an approximative analytical solution can be found after replacing the Yukawa potential by the Hulthen potential

$$V_H = -\alpha_D \; m_* \frac{e^{-m_* r}}{1 - e^{-m_* r}}. \tag{24}$$

Then, the Sommerfeld factor for direct annihilation is found to be [22], [23]

$$S_H = \frac{2\pi\alpha_D}{v_{\rm rel}} \frac{\sinh(\pi M_{\rm DM} v_{\rm rel}/m_*)}{\cosh(\pi M_{\rm DM} v_{\rm rel}/m_*) - \cosh(\pi\sqrt{M_{\rm DM}^2 v_{\rm rel}^2/m_*^2 - 4M_{\rm DM}\alpha_D/m_*})}. \tag{25}$$

For $m_V \gtrsim M_{\rm DM} v_{\rm rel}/2$, the Sommerfeld factor with the Hulthen potential S_H is suppressed with respect to the Sommerfeld factor with the Coulomb potential S_C except on resonances, which coincide with energies of bound states crossing zero. S_H is a good approximation off-resonance. The position of the first resonances of the Yukawa potential can be matched if $m_* = 1.68\; m_V$ [22] but the position of the higher resonances become more and more approximate.

B Unitary bound on cross-sections

Clarification of the computation of the unitary bound on $2 \to 2$ cross-section in [4] using results from [24].

B.1 Partial-wave expansion of the cross-section

We consider the scattering $1 + 2 \to 3 + 4$. The transition probability amplitude between the two asymptotic states $\langle i| S |f\rangle = \langle p_1, \lambda_1; p_2, \lambda_2| S |p_3, \lambda_3; p_4, \lambda_4\rangle$ can be decomposed as

$$S_{if} = \delta_{if} + i(2\pi)^4 \delta^{(4)}(P_i - P_f) T_{if} \tag{26}$$

where the matrix element T_{if} contains the non-trivial part. The cross-section reads

$$d\sigma(ab \to cd) = \frac{1}{2E_1 2E_2 v_{\rm rel}} d\,{\rm Lips}({\rm s}; {\rm P}_3, {\rm P}_4) \tag{27}$$

where

$$2E_1 2E_2\; v_{\rm rel} = \sqrt{(p_1 \cdot p_2)^2 - m_1 m_2} \equiv 2\lambda^{1/2}(s, m_1^2, m_2^2) \tag{28}$$

In the rest frame of particule a, we have $v_{\rm rel} = p_2/E_2 = v_2$ while in the center of mass frame we have

$$v_{\rm rel} = \frac{p\sqrt{s}}{E_1 E_2}. \tag{29}$$

We notice that the center of mass initial p_i and final p_f momenta can be expressed as

$$p_i = (4s)^{-1/2}\lambda^{1/2}(s, m_1^2, m_2^2), \quad p_f = (4s)^{-1/2}\lambda^{1/2}(s, m_3^2, m_4^2) \tag{30}$$

The Lorentz-invariant phase space is

$$d\,{\rm Lips}({\rm s}, {\rm P}_3, {\rm P}_4) = \frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}(2\pi)^4\; \delta(P_1 + P_2 - P_3 - P_4) \tag{31}$$

$$= \frac{1}{16\pi^2}\frac{1}{E_3 E4} p^2 dp\; d\Omega\; \delta(s^{1/2} - E_1 - E_2). \tag{32}$$

Now using

$$E = (m_3^2 + p^2)^{1/2} + (m_4^2 + p^2)^{1/2} \tag{33}$$

$$dE = (E_3^{-1} + E_4^{-1})\, p\, dp = E_3^{-1} E_4^{-1} E\, p\, dp, \tag{34}$$

the LIPS becomes

$$d\,\mathrm{Lips}(\mathrm{s}, \mathrm{P}_1, \mathrm{P}_2) = \frac{1}{16\pi^2} \frac{\mathrm{p}}{\mathrm{E}} \delta(\mathrm{s}^{1/2} - \mathrm{E}) \mathrm{dEd}\Omega \tag{35}$$

and

$$\boxed{d\,\mathrm{Lips}(\mathrm{s}, \mathrm{P}_1, \mathrm{P}_2) = \frac{1}{16\pi^2} \frac{\mathrm{p}}{\sqrt{\mathrm{s}}} \mathrm{d}\Omega} \tag{36}$$

Then the differential cross section reads

$$d\sigma(12 \to 34) = \frac{p_f}{4p_i s} \sum_f \left| \frac{T_{if}}{4\pi} \right|^2 d\Omega \tag{37}$$

$$= \frac{\pi p_f}{2p_i s} \sum_f \left| \frac{T_{if}}{4\pi} \right|^2 d\cos\theta \tag{38}$$

Now, we expand $T_{if}(\theta)$ in terms of Legendre polynomials of $\cos\theta$:

$$T_{if}(\theta) = 8\pi s^{1/2} \sum_{L=0}^{\infty} (2L+1) P_L(\cos\theta) T_{if,L}(s). \tag{39}$$

Using the orthogonality relation

$$\frac{1}{2} \int dx P_{L'}(x) P_L(x)(2L+1) = \delta_{LL'} \tag{40}$$

we obtain the cross section in terms of partial waves $\sigma = \sum \sigma_L$

$$\sigma_L = 4\pi(2L+1) \sum_f \frac{p_f}{p_i} \left| T_{if,L} \right|^2, \tag{41}$$

which averaged over polarizations becomes

$$\boxed{\sigma_L = \frac{4\pi(2L+1)}{(2s_1+1)(2s_2+1)} \sum_\lambda \sum_f \frac{p_f}{p_i} \left| T_{if,L} \right|^2.} \tag{42}$$

B.2 Unitarity of the partial-wave expansion

The unitarity means that the Hamiltionian should be Hermitian $H^\dagger = H$. Since the S-matrix is $S = e^{-iHt}$, it implies

$$S^\dagger S = 1 \tag{43}$$

and yields to

$$-i(2\pi)^4 \delta^{(4)}(P_i - P_f)(T_{if} - T_{fi}^*) = \sum_n \left(\prod_{k=1}^n \int \frac{d^3 p_k}{(2\pi)^3} \frac{1}{2E_k} \right) T_{ik} T_{fk}^* (2\pi)^4 \delta^{(4)}(P_i - P_k)(2\pi)^4 \delta^{(4)}(P_f - P_k), \tag{44}$$

where we used Eq.(26). We get the generalized optical theorem

$$\boxed{-i(T_{if} - T_{fi}^*) = \sum_n \left(\prod_{k=1}^n \int \frac{d^3 p_k}{(2\pi)^3} \frac{1}{2E_k} \right) T_{ik} T_{fk}^* (2\pi)^4 \delta^{(4)}(P_i - P_k)} \tag{45}$$

which holds order by order. Remark: taking $i = f = A$, we obtain

$$\mathrm{Im\ T(A \to A) = m_1 \sum_X \Gamma(A \to X) = m_1 \Gamma_{tot}} \tag{46}$$

if A is a 1-particle state and

$$\mathrm{Im\ T(A \to A) = 2E_{cm}\, p_{cm} \sum_X \sigma(A \to X)} \tag{47}$$

if A is a 2-particles state. In the case of the scattering $12 \to 34$, by using Eq.(36) we get

$$-i(T_{if}(\Omega) - T^*_{fi}(\Omega)) = \sum_k \int d\ \mathrm{Lips(s;k) T_{ik} T^*_{fk}} \tag{48}$$

$$= \sum_k \frac{1}{16\pi^2} \frac{p_k}{\sqrt{s}} \int d\Omega' T_{ik}(\Omega') T^*_{fk}(\Omega'') \tag{49}$$

where Ω is the solid angle between p_1 and p_3, Ω' is the solid angle between p_1 and p_k and Ω'' is the solid angle between p_3 and p_k. Please see Fig.(2) for a visual representation.

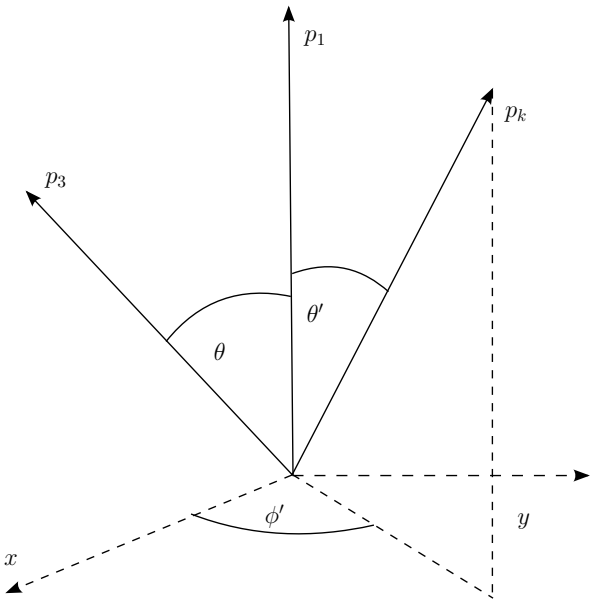


Figure 2: Solid angles Ω, Ω' and Ω'' which appear in the unitarity equation Eq.(49).

Now using the expansion in Legendre polynomials in Eq.(39), the following property of Legendre polynomials

$$\int d\phi^{'} P_L(\cos\theta'') = 2\pi P_L(\cos\theta) P_L(\cos\theta') \tag{50}$$

as well as the orthogonality relation in Eq.(40), we can write

$$-8\pi s^{1/2} \sum_L (2L+1) P_L(x) i(T_{if,L} - T^*_{fi,L}) \tag{51}$$

$$= \sum_k p_k\ 4s^{1/2} \int dx' \sum_{L'} (2L'+1) P_{L'}(x') T_{ik,L'} \sum_L (2L+1) 2\pi P_L(x) P_L(x') T^*_{fk,L} \tag{52}$$

$$= 16\pi s^{1/2} \sum_k p_k \sum_{L'} (2L'+1) P_{L'}(x) T_{ik,L'} T^*_{fk,L'}, \tag{53}$$

with $x = \cos\theta$ and $x' = \cos\theta'$. Finally, we get

$$T_{if,L} - T^*_{fi,L} = 2i \sum_k p_k \, T_{ik,L} T^*_{fk,L}. \tag{54}$$

Using matrix notation, this can be rewritten as

$$\boxed{T_L - T_L^\dagger = 2i T_L \tilde{p} T_L^\dagger} \tag{55}$$

with $\tilde{p} = \mathrm{diag}(...p_k...)$ where p_k is the 3-momentum of the intermediate state k. Defining $S_L = 1 + 2i\tilde{p}^{1/2} T_L \tilde{p}^{1/2}$, we see that the partial-wave unitarity can also be written $S_L S_L^\dagger = 1$ or

$$|S_{el,L}| + \sum_f |S_{i\neq f,L}|^2 = 1, \tag{56}$$

where $S_{el,L}$ stands for the elastic channel, $i = f$. If we define $S_{el,L} = \eta_L e^{2i\delta_L}$, where δ_L is a real phase shift and η_L is an inelaticity factor, $0 \leq \eta_L \leq 1$. Then $|S_{el,L}|^2 = \eta_L^2$, and $\sum_f |S_{i\neq f,L}|^2 = 1 - \eta_L^2$. Finally, using $T_{el,L} = (S_{el,L} - 1)/2ip$ and $T_{f\neq i,L} = S_{f\neq i,L}/2i(p_i p_f)^{1/2}$ and the standard formula for the unpolarized cross section in terms of partial waves $\sigma = \sum \sigma_L$, where σ_L is given in Eq.(42), we find

$$\sigma_{r,L} = 4\pi \frac{2L+1}{(2s_1+1)(2s_2+1)} \sum_\lambda \sum_{f\neq i} \frac{p_f}{p_i} |T_{if,L}|^2 \tag{57}$$

$$= \frac{\pi(2L+1)(1-\eta_L)}{p_i^2}. \tag{58}$$

Here $\sigma_{r,L}$ is the *reaction* cross section, that is, the total cross section minus the elastic piece. It has a maximum when $\eta_L = 0$, so we conclude that

$$\sigma_L(a + b \to c + d) \leq \pi(2L+1)/p_i^2. \tag{59}$$

In the early Universe,

$$p_i^2 = E^2 - m_X^2 = \frac{m_X^2 v_{\rm rel}^2}{4(1 - v_{\rm rel}^2/4)} \approx \frac{m_X^2 v_{\rm rel}^2}{4}, \tag{60}$$

where we used $v_i = p_i/E$ and $v_{\rm rel} = 2v_i$.
So $\sigma_L v_{\rm rel} \leq (\sigma_L)_{\rm max} \, v_{\rm rel}$ where

$$\boxed{(\sigma_L)_{\rm max} v_{\rm rel} \approx \frac{4\pi(2L+1)}{m_X^2 v_{\rm rel}}} \tag{61}$$

C Gamma-ray from DM annihilation

Let's compute the contribution to the detected gamma spectrum due to the Dark Matter annihilation in the Milky way. The number of DM pairs in the volume d^3r at position $\vec{r}$ in the MW is

$$d^3r \frac{1}{2} \left(\frac{\rho_{DM}(\vec{r})}{M_{\rm DM}} \right)^2 \tag{62}$$

where $\rho_{DM}(\vec{r})$ is the DM energy density at $\vec{r}$. We have assumed self-conjugate DM. In the opposit case where particles can only interact with anti-particles, the 1/2 must be replaced by 1/4.

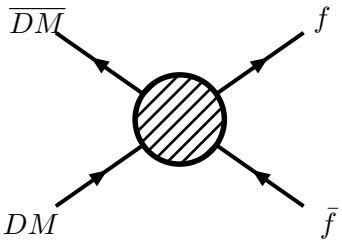


Figure 3: DM annihilation into SM

If we suppose that a pair of DM annihilates into a pair of SM particles f as shown in Fig.(3) with a cross-section $\langle \sigma v \rangle_f$, then multiplying the last expression by $\langle \sigma v \rangle_f$ gives the number of DM annihilations into f per second

$$d^3 r \frac{1}{2} \left(\frac{\rho_{DM}(\vec{r})}{M_{\rm DM}} \right)^2 \sum_f \langle \sigma v \rangle_f \,. \tag{63}$$

Now multiplying by the number of photons produced per second, in energy bin ΔE centered on E, per DM annihilation into SM particles f,

$$\Delta E \frac{dN_{f \to \gamma}}{dE}, \tag{64}$$

we get the number of photons produced in volume $d^3 r$, at position $\vec{r}$, in energy bin ΔE centered on E

$$d^3 r \frac{1}{2} \left(\frac{\rho_{DM}(\vec{r})}{M_{\rm DM}} \right)^2 \sum_f \langle \sigma v \rangle_f \, \Delta E \frac{dN_{f \to \gamma}}{dE}. \tag{65}$$

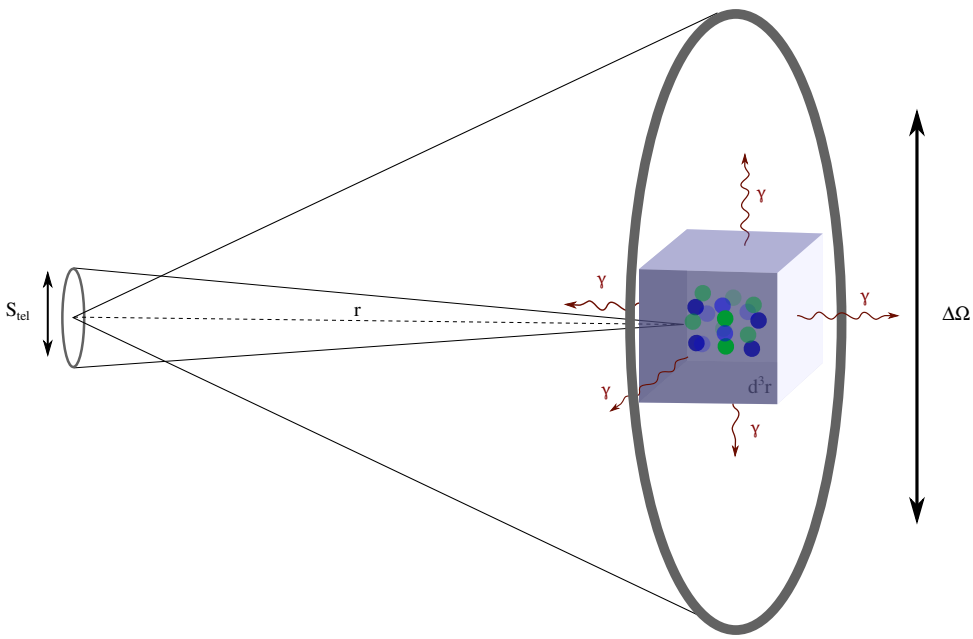


Figure 4: The detected γ-ray flux is the fraction $\frac{\Delta\Omega}{4\pi} \frac{S_{\rm tel}}{4\pi r^2}$ of the total emission.

However, as described in Fig.(4) a telescope with a surface $S_{\rm tel}$, receiving signals within a solid angle $\Delta\Omega$ and within an energy bin ΔE detects only a fraction

$$\frac{\Delta\Omega}{4\pi} \frac{S_{\rm tel}}{4\pi r^2} \tag{66}$$

of the total number of photons produced. So this telescope detects a number of photons per second which is

$$S_{tel}\Delta\Omega\Delta E\frac{d\phi_\gamma}{d\Omega dE} = \int d^3r\frac{1}{2}\left(\frac{\rho_{DM}(\vec{r})}{M_{\rm DM}}\right)^2\sum_f \langle\sigma v\rangle_f\,\Delta E\frac{dN_{f\to\gamma}}{dE}\frac{\Delta\Omega}{4\pi}\frac{S_{\rm tel}}{4\pi r^2}. \tag{67}$$

where ϕ_γ is the integrated γ flux in cm^2/s.
Finally the detected gamma flux per solid angle unit and per energy unit, assuming self-conjugate DM, is

$$\frac{d\phi_\gamma}{d\Omega dE} = \frac{1}{2}\frac{r_\odot}{4\pi}\left(\frac{\rho_\odot}{M_{\rm DM}}\right)^2 J_{\rm ann}\sum_f \langle\sigma v\rangle_f\,\frac{dN_{f\to\gamma}}{dE} \tag{68}$$

where

$$J_{\rm ann} = \int\frac{ds}{r_\odot}\left(\frac{\rho_{DM}(\vec{r})}{\rho_\odot}\right)^2 \tag{69}$$

is the so-called $J_{\rm ann}$ factor for annihilation, $r_\odot = 8.33$ kpc is the Earth distance to the GC and $\rho_\odot = 0.3$ GeV/cm^3 is the supposed DM energy density at the Earth position.
Note that a similar treatment for DM decay leads to

$$\frac{d\phi_\gamma}{d\Omega dE} = \frac{r_\odot}{4\pi}\left(\frac{\rho_\odot}{M_{\rm DM}}\right) J_{\rm dec}\sum_f \Gamma_f\frac{dN_{f\to\gamma}}{dE} \tag{70}$$

where

$$J_{\rm dec} = \int\frac{ds}{r_\odot}\left(\frac{\rho_{DM}(\vec{r})}{\rho_\odot}\right). \tag{71}$$

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