Gravitational Wave and Phase Transitions

Prof. Dr. Pedro Schwaller, Eric Madge August 30, 2019

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1. **Electroweak Phase Transition in SM + Dimension-6 Operator**
The electroweak phase transition in the SM has been shown to be a cross-over. To obtain a first-order transition we have to go beyond the SM. A simple way to achieve this is to add a dimension-six operator $|H|^6$ to the Higgs potential, i.e.

$$V(H) = \mu^2 H^\dagger H + \lambda \left(H^\dagger H\right)^2 + f^{-2} \left(H^\dagger H\right)^3 . \tag{1}$$

In the following we will determine the range of the parameter f that gives rise to a first-order transition. For simplicity we neglect all fermionic contributions to the effective Higgs potential apart from the top quark. The relevant part of the Lagrangian is given by

$$\mathcal{L} \supset D^\mu H^\dagger D_\mu H - V(H) + \left(y_t \overline{Q_L^3} \tilde{H} t_R + \text{h.c.}\right) , \tag{2}$$

where $D_\mu H = \left(\partial_\mu - ig\frac{\sigma^a}{2}W_\mu^a - ig'\frac{1}{2}B_\mu\right) H$, $Q_L^3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix}$, and $\tilde{H} = i\sigma_2 H$. We consider fluctuations around a constant background field ϕ, choosing ϕ to be in the real part of the neutral component, i.e. $H = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(\phi + h + iG^0\right) \end{pmatrix}$, where h is the physical Higgs field, and G^+ and G^0 are Goldstone bosons.

(a) Write the potential (1) as a function of the background field ϕ only, setting h, G^0 and G^+ to 0.

(b) We know experimentally that the Higgs acquires a vacuum expectation value $v = 246\,\text{GeV}$ and has a mass of $m_h = m_h(\phi = v) = 125\,\text{GeV}$. Express the potential parameters λ and μ^2 through v, m_h^2, and f. For which values of f is $\phi = v$ the global minimum (at $T = 0$)?

(c) Extract the background-field dependent masses of the gauge, Higgs and Goldstone bosons, and the top quark in Landau gauge ($\xi = 0$).

(d) The one-loop effective potential to leading order in $m^2(\phi)/T^2$ can be written as

$$V_{\text{eff}}(\phi, T) = \frac{C_2(T)}{2}\phi^2 + \frac{C_4(T)}{4}\phi^4 + \frac{C_6(T)}{6}\phi^6 , \tag{3}$$

where field-independent terms have been dropped. Show that the coefficients $C_i(T)$ are

$$C_2(T) = -\frac{m_h^2}{2} + \frac{3}{4}\frac{v^4}{f^2} + \frac{1}{4}\left(m_h^2 + M^2 - 3\frac{v^4}{f^2}\right)\frac{T^2}{v^2} , \tag{4}$$

$$C_4(T) = \frac{1}{2}\frac{m_h^2}{v^2} - \frac{3}{2}\frac{v^2}{f^2} + \frac{T^2}{f^2} , \tag{5}$$

$$C_6(T) = \frac{3}{4f^2} \tag{6}$$

where $m_Z = 91\,\text{GeV}$, $m_W = 80\,\text{GeV}$, and $m_t = 169\,\text{GeV}$ are the physical particle masses.

(e) For simplicity let us drop the $T^2\phi^4$ term in the following. For which values of f does the model exhibit a first-order phase transition? What are the corresponding critical temperatures?

2. **Bounce Solution**

The Euclidean action for tunneling from the false to the true vacuum phase is given by [1]

$$S_{E,d}[\phi_b] = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\partial_\mu \phi_b\right)^2 + V(\phi_b)\right], \tag{7}$$

where $d = 3$ (4) for thermal (quantum) tunneling, and ϕ_b is the $SO(d)$ symmetric bounce solution to the differential equation

$$\ddot{\phi}_b + \frac{d-1}{r}\dot{\phi}_b = V'(\phi_b) \tag{8}$$

with boundary conditions $\lim_{r\to\infty} \phi_b(r) = \phi_+$ and $\dot{\phi}(0) = 0$. Here, ϕ_+ and ϕ_- denote the positions of the minima of V with $V(\phi_-) < V(\phi_+)$. We assume $V(\phi_+) = 0$ in the following.

One way to calculate the Euclidean tunneling action that has been proposed recently [2] is by defining the tunneling Potential $V_t(\phi) = V(\phi) - \dot{\phi}_b$. It satisfies the differential equation

$$\left(V_t'\right)^2 = \frac{d-1}{d}\left[V' V_t' + 2\left(V_t - V\right) V_t''\right] \tag{9}$$

with boundary conditions $V_t(\phi_+) = V(\phi_+)$ and $V_t(\phi_0) = V(\phi_0)$, where ϕ_0 is the point to which the field tunnels. It can be shown that $V_t(\phi)$ is monotonically decreasing between ϕ_+ and ϕ_0 with $V_t(\phi) \leq V(\phi)$.

In terms of V_t the tunneling action becomes

$$S_{E,d}[V_t] = \frac{(d-1)^{d-1}(2\pi)^{d/2}}{\Gamma(1+d/2)} \int_{\phi_+}^{\phi_0} \mathrm{d}\phi \frac{\left(V - V_t\right)^{d/2}}{|V_t'|^{d-1}}. \tag{10}$$

We can now obtain an approximation of S_E by approximating V_t for a given ϕ_0 and then minimizing $S_E[V_t]$ with respect to ϕ_0.

(a) Approximate $V_t(\phi)$ as a fourth-order polynomial

$$V_{t,a}(\phi) = a_1\phi + a_2\phi(\phi - \phi_0) + a_3\phi(\phi-\phi_0)^2 + a_4\phi^2(\phi-\phi_0)^2. \tag{11}$$

Assume that $\phi_+ = 0$ ($V(\phi_+) = 0$) and subsequently determine the coefficients a_i by requiring that

i. the boundary conditions $V_t(\phi_+) = V(\phi_+)$ and $V_t(\phi_0) = V(\phi_0)$ are satisfied.

ii. the differential equation (9) is satisfied at ϕ_0.

iii. (9) is satisfied at ϕ_0.

iv. (9) is satisfied at ϕ_T, the maximum of $V(\phi)$.

(b) Implement the method described above and use it to calculate the thermal tunneling action ($d = 3$) for the model considered in exercise 1 for $f = 600\,\mathrm{GeV}$ and $T = 50\,\mathrm{GeV}$. *Note:* $\phi_r < \phi_0 < \phi_-$, where ϕ_- is the position of the broken minimum and $\phi_T < \phi_r < \phi_-$ is a root of V.

(c) The nucleation temperature T_n is approximately given by $\frac{S_{E,3}(T_n)}{T_n} = 140$. Calculate the nucleation temperature for $f = 600\,\mathrm{GeV}$.

3. **Gravitational Wave Spectrum**

(a) Plot the GW spectrum $h^2\Omega_{\rm GW}(f)$ ($\Omega_{\rm GW}(f) = \frac{1}{\rho_{\rm crit}}\frac{{\rm d}\rho_{\rm GW}}{{\rm d}\log f}$) from a cosmological phase transition with $T_* = 200\,{\rm GeV}$, $\alpha = 0.1$, $\beta/H_* = 10$, and $g_* = 106.75$ for the case of a transition in vacuum and in a thermal plasma. How does the spectrum change when you change these parameters?

(b) The signal-to-noise ratio of a SGWB in LISA is given by

$$\mathrm{SNR} = \sqrt{\mathcal{T}\int_{f_{\rm min}}^{f_{\rm max}} {\rm d}f \left(\frac{h^2\Omega_{\rm GW}(f)}{h^2\Omega_n(f)}\right)^2}\,, \tag{12}$$

where $\mathcal{T}$ is the observation time, $h^2\Omega_n(f)$ is the noise density parameter of LISA, and $(f_{\rm min}, f_{\rm max})$ is the frequency range accessible to LISA. Calculate the SNR for the spectra in (a) assuming a duration of $\mathcal{T} = 3\,{\rm yrs}$.

(c) A GW signal is detectable if it produces an SNR $>$ SNR$_{\rm thr}$. For LISA, SNR$_{\rm thr} = 10$. Consider a power-law spectrum $h^2\Omega(f) = h^2\Omega_p\left(\frac{f}{f_0}\right)^p$ with $f_0 = 1\,{\rm mHz}$. Compute the minimal detectable amplitude $h^2\Omega_p^{\rm thr}$ for $p \in \{-8, -7, \ldots, 8\}$.

(d) To graphically represent the sensitivity to SGWBs one uses the so-called power-law sensitivity [3]. It is given by the envelope of the minimal detectable power-law signals.

$$h^2\Omega_{\rm PLS} = \max_p \left[h^2\Omega_p^{\rm thr}\left(\frac{f}{f_0}\right)^p\right] \tag{13}$$

Plot the PLS along with the spectra from (a).

Formulary

Effective Potential

The 1-loop contribution to the effective potential is given by

$$V_1(\phi) = \sum_i \pm \frac{n_i m_i^4(\phi)}{64\pi^2}\left(\log\frac{m_i^2(\phi)}{\mu_R^2} - C_i\right) + \sum_i \frac{n_i T^4}{2\pi^2} J_\mp\left(\frac{m_i^2(\phi)}{T^2}\right), \tag{14}$$

where the first sum is the zero-temperature Coleman-Weinberg potential [4] in $\overline{\text{MS}}$ renormalization, and the second sum contains the thermal corrections [5]. The sums run over all species coupled to ϕ, and n_i and $m_i^2(\phi)$ are the number of degrees of freedom and the field-dependent squared masses of the species i. The upper (lower) sign corresponds to bosons (fermions). μ_R is the renormalization scale and $C_i = \frac{3}{2}\left(\frac{5}{6}\right)$ for scalars and fermions (gauge bosons).
The thermal one-loop functions

$$J_\pm(x^2) = \mp\int_0^\infty \mathrm{d}k\, k^2 \log\left(1 \pm e^{-\sqrt{k^2+x^2}}\right) \tag{15}$$

can be expanded for high temperatures ($x^2 \ll 1$) as

$$J_+(x^2) = -\frac{7\pi}{360} + \frac{\pi^2}{24}x^2 + \frac{x^4}{32}\log\frac{x^2}{a_+} + \mathcal{O}\left(x^6\right), \tag{16}$$

$$J_-(x^2) = -\frac{\pi^2}{45} + \frac{\pi^2}{12}x^2 - \frac{\pi}{6}\left(x^2\right)^{\frac{3}{2}} - \frac{x^4}{32}\log\frac{x^2}{a_-} + \mathcal{O}\left(x^6\right), \tag{17}$$

where $a_+ = \pi^2 \exp\left(\frac{3}{2} - 2\gamma_E\right)$ and $a_- = 16\pi^2 \exp\left(\frac{3}{2} - 2\gamma_E\right)$.

GW Spectrum

A cosmological first-order phase transition can be characterized by three parameters:[1]

- the transition temperature $T_* \simeq T_n$
- the transition strength $\alpha \simeq \frac{\Delta V}{\rho_*^{\text{rad}}}$
- the transition time scale β^{-1} with $\frac{\beta}{H_*} = \left[T\frac{\mathrm{d}}{\mathrm{d}T}\frac{S_3(T)}{T}\right]_{T=T_*}$

where T_n is the nucleation temperature, ΔV is the potential difference between the two minima, ρ_*^{rad} and H_* are the energy density of the Universe and the Hubble rate at T_*, and S_3 is the bounce action.
A phase transition can generate gravitational waves via three mechanisms: the collision of bubbles of the broken vacuum, sound waves, and turbulence. The corresponding graviational wave spectra are given by [6–11]

$$h^2\Omega_{\text{col}}(f) = 0.028\,\mathcal{R}\left(\frac{H_*}{\beta}\right)^2\left(\frac{\kappa_{\text{col}}\alpha}{1+\alpha}\right)^2 S_{\text{col}}(f), \tag{18}$$

$$h^2\Omega_{\text{sw}}(f) = 0.29\,\mathcal{R}\left(\frac{H_*}{\beta}\right)(H_*\tau_{\text{sh}})\left(\frac{\kappa_{\text{sw}}\alpha}{1+\alpha}\right)^2 S_{\text{sw}}(f), \tag{19}$$

$$h^2\Omega_{\text{turb}}(f) = 20\,\mathcal{R}\left(\frac{H_*}{\beta}\right)(1 - H_*\tau_{\text{sh}})\left(\frac{\kappa_{\text{sw}}\alpha}{1+\alpha}\right)^{\frac{3}{2}} S_{\text{turb}}(f), \tag{20}$$

[1] In principle there is a fourth parameter: the bubble wall velocity v_w. We here take $v_w = 1$.

with the spectral shapes

$$S_{\rm col}(f) = \left(\frac{f}{f_{\rm col}}\right)^3 \left[\frac{4.51}{1.51 + 3\,(f/f_{\rm col})^{2.07}}\right]^{2.18} , \tag{21}$$

$$S_{\rm sw}(f) = \left(\frac{f}{f_{\rm sw}}\right)^3 \left[\frac{7}{3 + 4\,(f/f_{\rm sw})^2}\right]^{\frac{7}{2}} , \tag{22}$$

$$S_{\rm turb}(f) = \left(\frac{f}{f_{\rm turb}}\right)^3 \left[\frac{1}{1 + (f/f_{\rm turb})}\right]^{\frac{11}{3}} \frac{1}{1 + 8\pi f/h_*} , \tag{23}$$

and peak frequencies

$$f_{\rm col} = 0.17\, h_* \left(\frac{\beta}{H_*}\right) , \qquad f_{\rm sw} = 0.54\, h_* \left(\frac{\beta}{H_*}\right) , \qquad f_{\rm turb} = \frac{3.5}{2}\, h_* \left(\frac{\beta}{H_*}\right) . \tag{24}$$

The red-shifting factor $\mathcal{R}$ of the density parameter and the Hubble rate at the phase transition red-shifted to today, h_*, are

$$\mathcal{R} = 1.67 \times 10^{-5} \left(\frac{g_*}{100}\right)^{-\frac{1}{3}} \qquad \text{and} \qquad h_* = 16.5\,\mu\text{Hz} \left(\frac{T_*}{100\,\text{GeV}}\right) \left(\frac{g_*}{100}\right)^{\frac{1}{6}} . \tag{25}$$

For a phase transition in vacuum, the efficiency factors κ can be approximated as $\kappa_{\rm col} = 1$ and $\kappa_{\rm sw} = 0$. For a transition in a thermal plasma we can use $\kappa_{\rm col} = 0$ and [12]

$$\kappa_{\rm sw} = \frac{\alpha}{0.73 + 0.083\sqrt{\alpha} + \alpha} . \tag{26}$$

The shock time $\tau_{\rm sh}$ in the plasma is [8]

$$H_*\tau_{\rm sh} = \min\left[1,\ \frac{H_* R_*}{\bar{U}_f}\right] \qquad \text{with} \quad H_* R_* = (8\pi)^{\frac{1}{3}} \left(\frac{\beta}{H_*}\right)^{-1} \quad \text{and} \quad \bar{U}_f^2 = \frac{3}{4}\frac{\kappa_{\rm sw}\alpha}{1+\alpha} . \tag{27}$$

LISA Sensitivity

The power spectral density (PSD) noise of LISA is given by [13]

$$S_n(f) = \frac{10}{3} \left\{ \frac{5.76 \times 10^{-48}\,\text{Hz}^3}{(2\pi f)^4} \left[1 + \left(\frac{0.4\,\text{mHz}}{f}\right)^2\right] + \frac{3.6 \times 10^{-41}}{1\,\text{Hz}} \right\} \left[1 + \left(\frac{f}{25\,\text{mHz}}\right)^2\right] \tag{28}$$

in the frequency window $3 \times 10^{-5}\,\text{Hz} < f < 0.5\,\text{Hz}$. The corresponding density parameter is ($H_{100} = 100\,\text{km}\,\text{Mpc}^{-1}\text{s}^{-1}$)

$$h^2\Omega_n(f) = \frac{4\pi^2}{3H_{100}^2} f^3 S_n(f) . \tag{29}$$

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