# Gravitational Waves

General relativity: $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

Equation of motion for the metric tensor $g_{\mu\nu}$ and the energy momentum tensor $T_{\mu\nu}$.

$G_{\mu\nu}$: combination of $g_{\mu\nu}$, $\partial_\mu g_{\nu\rho}$, $\partial_\mu \partial_\nu g_{\rho\sigma}$.

$$8\pi G_N \equiv \frac{1}{M_{pl}^2}$$

Example from Cosmology: $3H^2 M_{pl}^2 = \rho_{tot}$ for FRW Cosmologies

Now, expand around small deviations from vacuum (Minkowski)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$|h_{\mu\nu}| \ll 1 \quad \forall\, \mu,\nu$$

Note: With non-carthesian coords, the $|h_{\mu\nu}| \ll 1$ constraint might need to be modified

$\rightsquigarrow$ Find $\quad \Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

$h = \eta^{\mu\nu} h_{\mu\nu}$

Note: Here we skip many details

- $h_{\mu\nu}$ also describes other perturbations like density fluctuations, weak fields etc → pick out the propagating part

- gauge invariance
  ↳ some historic confusion of whether GW's exist (i.e. gauge independent?)

Vacuum solutions:

$$h^{TT}_{\mu\nu} = C_{\mu\nu} \, e^{i k_\sigma x^\sigma}$$

↑ 2 polarisations

plane waves, $k_\mu k^\mu = 0$

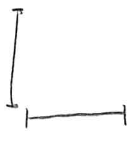
from $\Box h_{\mu\nu} = 0$

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Effect on matter from geodesic deviation eqn (Carroll book etc)

$C_+$

and other pol. rotated by 45% $C_\times$

↳ Detector:  measure length of perpendicular rods

Sources: Green's fn of d'Alambert $\Box_x G(x-y) = \delta^4(x-y)$

$$\Rightarrow \quad \bar{h}_{\mu\nu}(x) = -16\pi G \int d^4y \, G(x-y) \, T_{\mu\nu}(y)$$

Now for the case of a single, distant source:

$$\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{r} \frac{d^2}{dt^2} I_{ij}(t_r)$$

$$\hookrightarrow \quad I_{ij}(t) = \int y^i y^j \, T^{00}(t,\vec{y}) \, d^3\vec{y}$$

quadrupole moment

In simple words: For a large signal, need something big ($\sim T^{00} = M$) to be accellerated or changing quickly ($\sim \frac{d^2}{dt^2}$)

- Binary systems

  - BH BH binary, in the last seconds the objects become close to relativistic

  - Binary pulsar (Hulse-Taylor)
    spin down of system due to grav. radiation observable

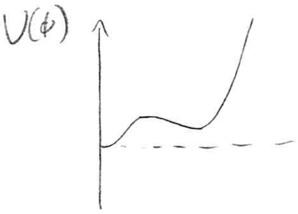
- Cosmological sources

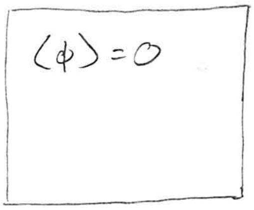
  - Inflation
  - Cosmic strings
  - Phase transitions (1st order)
  - Axions (see 1811.01950)

# GW's from PT's

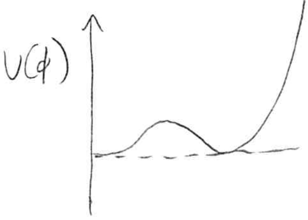
Potential | Universe (1 Hubble volume)

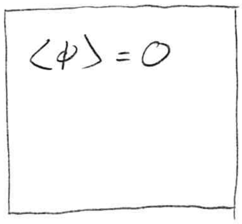
$T > T_c$



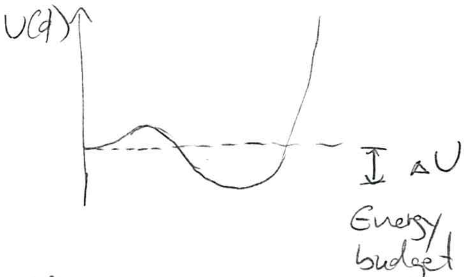



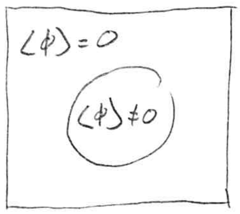

$T = T_c$



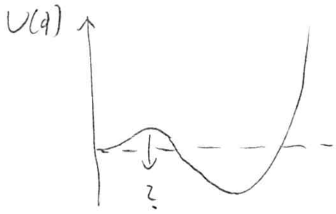



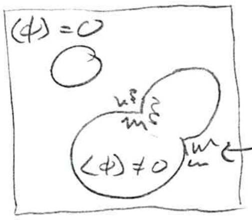

$T = T_n < T_c$



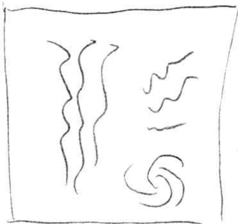

$T < T_n$




GW production

$T \ll T_n$



sound waves,
turbulence in plasma
more GW production

$T_c$ from $U(\phi, T)$. How to find $T_N$ ?

Vacuum decay rate $\quad \Gamma(T) \propto T^4 e^{-S_3/T}$

$S_3(T)$ : Action of the $O(3)$ symmetric tunneling "bounce" solution

$\Leftrightarrow$ How much energy is needed to cross the barrier

The <u>nucleation temperature</u> $T_N$ is defined by the requirement that one bubble per Hubble volume should be nucleated:

$$\frac{\Gamma(T)}{H^4(T)} \overset{!}{=} 1$$

Now $\quad H \sim \frac{T^2}{M_{pl}} \Rightarrow \frac{\Gamma}{H^4} \sim \frac{M_{pl}^4}{T^4} e^{-S_3/T} \Rightarrow$

$$\frac{S_3}{T} \sim -\mathrm{Log}\left(\frac{T^4}{M_{pl}^4}\right) \sim 140 \qquad \text{for } T \sim \text{weak scale}$$

Technical details: $\qquad$ Coleman, PRD 15, 10 p. 2929 (1977)
(see. Exercises)

$$S_3(\phi_b) = \int d^3x \left(\frac{1}{2}(\nabla\phi_b)^2 + U(\phi_b)\right)$$

where $\phi_b$ is the <u>bounce</u> solution, i.e. solves $\quad \frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = V'(\varphi)$

with $\varphi \to 0$ at $r \to \infty$, $\frac{d\varphi}{dr} = 0$ at $r = 0$. $\quad$ (and $U = T(\varphi, T)$)

How fast does the transition complete?

$$\beta \equiv - \frac{dS}{dt}\Big|_{t_N} \quad \rightsquigarrow \quad \frac{\beta}{H}\Big|_{T_N} = T_N \frac{dS}{dT}\Big|_{T_N}$$

For large $\beta$, $\frac{\Gamma}{H}$ increases rapidly and the PT is fast

Energy budget: $\alpha \approx \frac{\Delta U}{\rho_{tot}} = \frac{\text{vacuum energy}}{\text{total energy}}$

Bubble wall speed ... difficult. Most PT's of interest have $v_w \to 1$.

Nucleation temperature $T_N \sim \langle \phi \rangle$ !

↳ Caveat: For slow, supercooled, vacuum dominated transitions, the PT might complete later ...

How to obtain the GW signal?

Difficult, requires numerical simulations (summary e.g. 1512.06239)

Qualitative:

Peak frequency at time of emission $f_* \sim \frac{1}{\lambda_*}$ ← wavelength

Characteristic length scale = bubble radius at time of collision

$$\lambda_* \sim \frac{1}{H_*} \left(\frac{H_*}{\beta}\right) \cdot \overset{\approx 1}{v_w}$$

↑ size of Hubble patch; $\left(\frac{H_*}{\beta}\right)$ ← how fast is the transition

Now redshift:

$$f_0 \equiv f_{today} = \frac{a_*}{a_0} \cdot f_* \approx \frac{T_0}{T_*} \cdot f_* \qquad \text{(entropy conservation)}$$

$$= \frac{T_0}{T_*} \cdot \frac{1}{H_*} \cdot \left(\frac{\beta}{H_*}\right)$$

$$\approx \frac{T_0}{M_{pl}} \cdot T_* \cdot \left(\frac{\beta}{H_*}\right)$$

$T_0 \sim 3K \sim 10^{-4}\,eV$

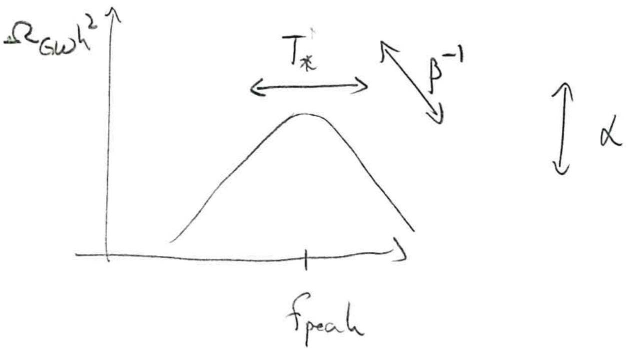
$M_{pl} \sim 10^{18}\,GeV$

$$T_* = \frac{T_*}{100\,GeV} \cdot 100\,GeV \cdot \frac{s}{s}$$

$$= \frac{T_*}{100\,GeV} \cdot 10^{26}\,Hz$$

$$\approx 10^{-5}\,Hz \cdot \left(\frac{T_*}{100\,GeV}\right) \cdot \left(\frac{\beta}{H_*}\right)$$

Signal shape:




What detector?

$10^{-3}\,Hz \rightarrow \lambda \sim 10^{10}\,m \sim 10^7\,km = 10\,Mkm$

↳ space → LISA

Sensitivity → SNR