Analytic Thin Wall Decay Rate

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CONTENT

I.) Introduction & Motivation

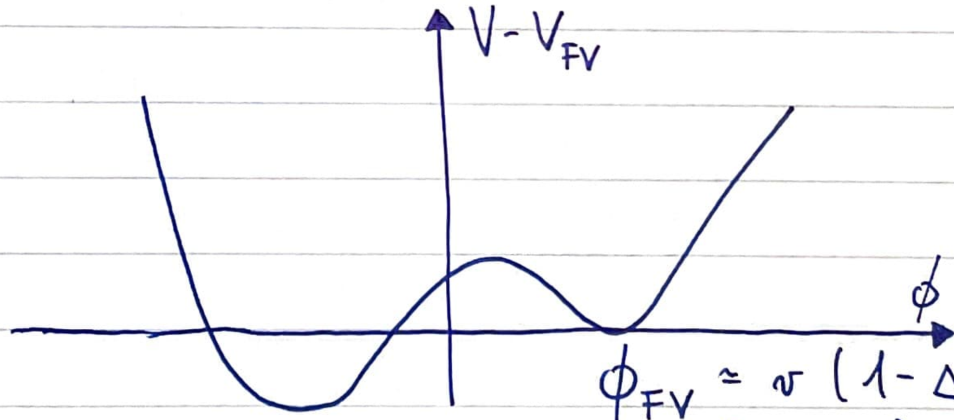
QFT / TFT ground states may not be stable.

Benchmark: real scalar field ϕ, singlet:

$$V = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 (\phi - v).$$

$\Delta = 0$: two degenerate minima $\langle \phi \rangle = \pm v$, no tunneling

$\Delta \gtrsim 0$: thin wall (TW) limit, FV is on the right.



$\phi_{FV} \simeq v\left(1 - \Delta - \frac{3}{2}\Delta^2 + \dots\right)$

$\phi_{TV} \simeq v\left(-1 - \Delta + \frac{3}{2}\Delta^2 + \dots\right)$

- at $\Delta = \frac{1}{3\sqrt{3}} \sim 0.19$, the FV disappears, no tunneling
- we assume $\lambda \ll 1$ & $\Delta \ll 1$ for perturbativity,
- we keep D general for as long as possible.

Physics Motivation

- stability at $T=0$: SM [Weinberg, Frampton ... ... Isidori, Rudolfi, Strumia, de Grassi et al., Andreassen et al., ...]
- meta-stability in BSM, precise limits on bi-quartics,
- phase transitions at $T>0$
  - production of : defects, BH, DM, GWs, B-fields, ...

History of $\Gamma = A e^{-B}$

- early works by Langer and Kobzarev et al.
- foundational works by Coleman et al.
  - Coleman on the bounce '77
  - Callan, Coleman on corrections '77

  } FLAT space-time, $T=0$

  - Coleman, De Luccia incl. gravity '79

II.) SETUP

Γ ... decay rate for nucleation

V ... volume

Coleman & Callan

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi}\right)^{D/2} \left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{FV}}\right|^{-\frac{1}{2}} \boxed{e^{-\frac{S}{\hbar} - S_{ct}}}$$

· dominant semi-classical when $\hbar \to 0$

· S is the Euclidean bounce action

MINKOWSKI $\quad \mathcal{L} = \int d^D x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right)$

$\Downarrow$

EUCLIDEAN $\quad S = \int d^D x_E \left(\frac{1}{2}\partial\phi^2 + V(\phi)\right)$

$\Downarrow$

O(D) SYMMETRIC $\quad S = \Omega \int_0^\infty d\rho\, \rho^{D-1}\left(\frac{1}{2}\dot\phi^2 + V - V_{FV}\right)$

$\hookrightarrow$ * proof by Coleman, Glaser & Martin '78

$\hookrightarrow$ multifields by Blum et al. '16 prime = removed zeroes

action counterterms

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi}\right)^{D/2} \left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{FV}} \right|^{-1/2} e^{-\frac{S}{\hbar} - S_{ct}}$$

dimensionless correction from zero removal, collective coordinates

the functional determinant

- S is the bounce action, calculated with renormalized parameters that run $\lambda_R = \lambda_R(\mu)$.
- S_{ct} comes from integrating the counter-term in dim-reg, we get the $\frac{1}{\varepsilon}$ poles $\times$ # numbers.
- $\mathcal{O}$ is the fluctuation operator, 2^{nd} derivative of $S[\phi]$

$$S[\phi] = S[\bar\phi] + \underbrace{\frac{\delta S}{\delta \phi}[\bar\phi]}_{=0} \delta\phi + \overbrace{\frac{\delta^2 S}{\delta \phi^2}}^{\mathcal{O}}[\bar\phi]\, \delta\phi^2$$

bounce action

$\bar\phi$ extremizes S, hence $\frac{dS}{d\phi} = 0$

first corrections, one loop

$$\mathcal{O} = -\partial_\mu \partial^\mu + V^{(2)} \quad , \quad V^{(n)} = \frac{dV^{(n)}}{d\phi^{(n)}}$$

- $\det \mathcal{O} \propto \prod \lambda_i$, $\mathcal{O}\Psi_i = \lambda_i \Psi_i$, each symmetry brings a zero λ (translations, internal global & local)

- the $\det' \mathcal{O}$ will be infinite and μ-dependent.

The $\frac{1}{\varepsilon}$ poles cancel with S_{ct}, the logs terms cancel with $S_R(\mu)$ $\Rightarrow$ Γ finite and μ-independent.

- Φ, the bouncer is $O(D)$ symmetric, only a function of the Euclidean radius ρ.

$\Downarrow$

$\mathcal{O}$ is symmetric, radially separable in $\mathcal{O}_\ell$

$$d_\ell = \frac{(2\ell+D-2)(\ell+D-3)!}{\ell!\,(D-2)!}, \quad \begin{matrix} d_0 = 1 \\ d_1 = D \end{matrix}$$

degeneracy in ℓ $\quad$ $\ell=0$ ... a single spherically symmetric mode (bubble)

$\ell=1$ ... D-translational modes that give zeroes and have to be removed

$$\boxed{\text{FINAL RESULT}}$$

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi} \frac{12}{\ell^{D-1}} \lambda v^2 \right)^{\frac{D}{2}} e^{-S - \frac{1}{\delta^{D-1}} \begin{cases} \frac{30 + 9 \ln 3}{54} , & D=3 \\ \frac{45 - 4\pi\sqrt{3}}{132} , & D=4 \end{cases}}$$

* This is done in the $\overline{MS}$ scheme with $\mu_0 = \sqrt{\lambda}\, v$.

III.) BOUNCE

III. i) Bounce action

III. ii) Higher orders

III. iii) Counterterms and running

III. i) Bounce action

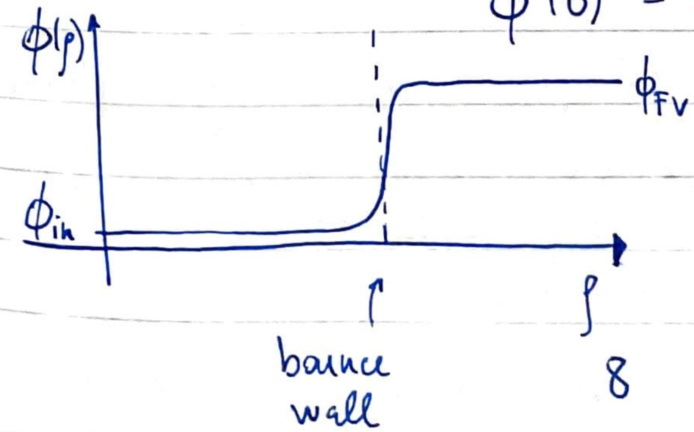
$$S = \int_D \frac{1}{2}\dot{\phi}^2 + V - V_{FV} \,, \quad \int_D = \Omega \int_0^\rho d\rho\, \rho^{D-1}$$

$$\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

- $\rho^2 = t^2 + x_i^2$
- Extremize $S \Rightarrow \bar{\phi}$ (I will call it ϕ)

  via E.L. $\Rightarrow$ $\ddot{\phi} + \frac{D-1}{\rho}\dot{\phi} = \frac{dV}{d\phi}$, $\dot{\phi}(0,\infty) = 0$

  $\phi(0) = \phi_{in}$, $\phi(\infty) = \phi_{FV}$



- wall $\to \infty$ when $\Delta \to 0$, $S \to \infty$, $\Gamma \sim e^{-S} \to 0$, $\tau \to \infty$ = NO TUNNELING.

THIN WALL EXPANSION

- We expand Φ, S and $\det \sigma$ in powers of $\Delta \ll 1$.

  This is a <u>systematic</u> approach for any D and separates all orders Δ^n, e.g. cancellations of $\frac{1}{\varepsilon}$, $\log\mu$ happen $\forall \Delta^n$.

- Dimensionless fields: $\phi = v \varphi$,

  radius: $z = \sqrt{\lambda}\, v \rho - r$.

- [short sidenote on units]

@ tree level $[\phi] = [v] = \frac{D}{2} - 1$, $[\lambda] = 4 - D$, $[\Delta] = 0$

then $[\sqrt{\lambda}\, v] = 2 - \cancel{\frac{D}{2}} + \cancel{\frac{D}{2}} - 1 = 1$ ... mass parameter

actually $V''_{FV} = V^{(2)}(\phi_{FV}) \simeq \lambda v^2 (1 + \mathcal{O}(\Delta))$

- TW perturbation expansion

$$\varphi = \sum \varphi_n \Delta^n \quad , \quad r = \frac{1}{\Delta} \sum r_n \Delta^n$$

LEADING ORDER

$\varphi = \varphi_0 \,, \qquad r = \frac{1}{\Delta} r_0 \xrightarrow{\Delta \to 0} \infty$

$V_0 \simeq \frac{\lambda}{8} (\phi^2 - v^2)^2$

- The solution to the bounce equation at $n = 0$ is

$$\varphi_0 = \operatorname{th}\left(\frac{z}{2}\right) \,, \quad \text{symmetric in } z \in [-r, \infty] \quad \xrightarrow{\Delta \to 0} \; -\infty$$

- The position of the bounce r (or r_0) is still undefined. We need to include the $n=1$ term in the potential $V_0 + \Delta v^3 (\phi - v)$.

This creates a tension between surface and volume terms and fixes

$$r_0 = \frac{D-1}{3}$$

- At this order, the action is given by

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \left(\frac{D-1}{3}\right)^{D-1} \frac{2}{3D}$$

- It satisfies Derrick's theorem : $(D-2) T = -D V$.

III. ii) HIGHER ORDER CORRECTIONS

- To proceed to higher $\Delta^{n>0}$ orders, we have to expand the bounce equation and the action S.
- In a sense, TW is a double expansion:

a) it expands $\frac{dV}{d\phi}$ in $\varphi = \varphi_0 + \Delta\varphi_1 + \Delta^2\varphi_2 + \ldots$

b) it expands $\frac{D-1}{\rho}\frac{d\phi}{d\rho}$ when $\sqrt{\lambda}\, v\, \rho = z + r$
$= z + \frac{1}{\Delta}(r_0 + r_1\Delta + \ldots)$

a) is a field expansion of $V^{(n)}$ and b) is the radius expansion with "friction" corrections

- Going further to higher orders, we get

$$\varphi_1 = -1 \quad (\text{even in } z), \qquad r_1 = 0$$

$$\varphi_2 = \frac{3}{4(D-1)}\frac{1}{\cosh^2\frac{z}{2}}\Big(\big(2-D-2(4+\cosh z)\ln(1+e^z)\big)\sinh z - z\big(D-e^z(4+\sinh z)\big) + 3\big(\mathrm{Li}_2(e^z) - \mathrm{Li}_2(e^{-z})\big)\Big)$$

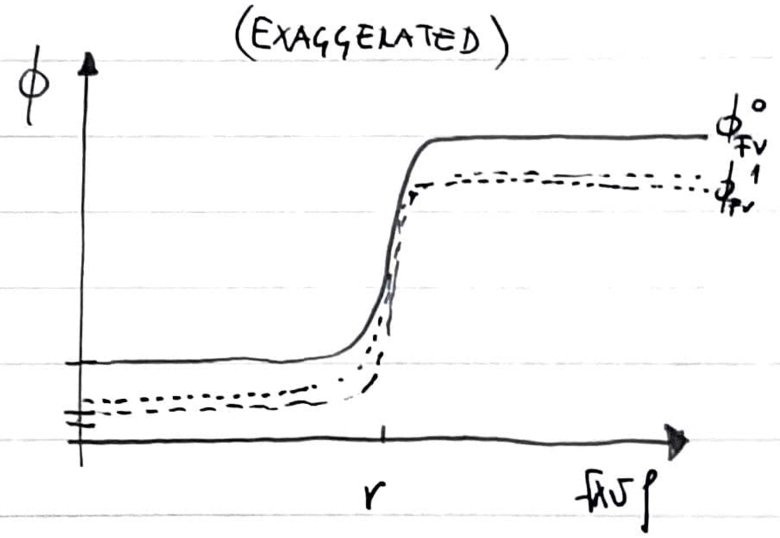
(odd in z)

- and the Δ^2 correction to the radius is

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}$$

- This constructions of the bounce matches the TV / FV at the ρ extrema

$$\left.\begin{aligned} \varphi_0(z \to \pm\infty) &= \pm 1, \\ \varphi_1(z \to \pm\infty) &= -1, \\ \varphi_2(z \to \pm\infty) &= \mp \frac{3}{2}. \end{aligned}\right\} \Rightarrow$$



- With Δ corrections, the bounce shifts (ϕ_1) and changes shape (ϕ_2). Its position remains the same at $n=1$, but shifts with r_2.

- Derrick's theorem still holds at $n=2$

$$(D-2)(T_0 + \Delta^2 T_2 + \dots) = -D(V_0 + \Delta^2 V_2 + \dots)$$

- The Euclidean action gets corrected @ Δ^2:

$$S = S_0\left(1 + \Delta^2\left(\overbrace{\frac{1+D(25-8D-3\pi^2)}{2(D-1)}}^{\text{NEW!!}}\right)\right)$$

(turns out r_2 and φ_2 don't really enter)

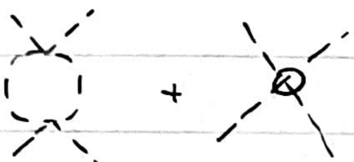
- Subsequent correction is $\mathcal{O}(\Delta^4)$, so this result is already quite precise; $\Delta \lessapprox \frac{1}{3\sqrt{3}} \sim 0.19$ anyways.

- It also serves as an estimate for the upper bound on Δ. For large D: $1 - \Delta^2 4D$; $\Delta \lesssim \frac{1}{2\sqrt{D}}$.

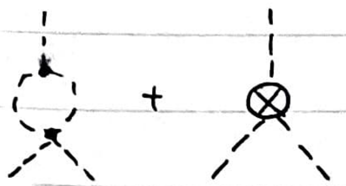
III iii) Counter-terms and running (D=4 here)

• At one loop, we renormalize with c.t.s

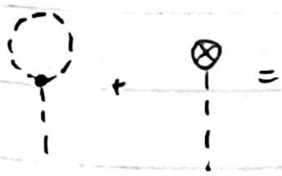
$$V_{ct} = \frac{1}{2}\delta_{m^2}\phi^2 + \frac{1}{4}\delta_\lambda \phi^4 \,, \qquad V^{(n)} = \frac{d^n V}{d\phi^n}(\langle\phi\rangle),$$

4P:  $= 0 \quad \Rightarrow \quad \delta_\lambda = \frac{1}{(4\pi)^2 2\varepsilon} V^{(4)2}$,

The δ_λ removes the $\frac{1}{\varepsilon}$ pole of the 3P as well:

 $= 0$ iff $\underbrace{V^{(3)} = \langle\phi\rangle V^{(4)}}_{\text{automatic in } \lambda\phi^4}$,

• The δ_{m^2} is fixed by the tadpole 1P

 $= 0 \qquad \delta_{m^2} = \frac{1}{(4\pi)^2\varepsilon} V^{(4)}\left(V^{(2)} - \frac{1}{2} V^{(4)}\langle\phi\rangle^2\right)$,

and works to remove the 2P pole

 $= 0$. ✓

- All of this works independently of Δ, $\delta_\Delta = 0$ and the Δ coupling does not run!

COUNTERTERM FOR THE ACTION : S_{ct}

- Remember $\frac{\Gamma}{V}$ contains the S_{ct}, since the $\frac{1}{\varepsilon}$ enter only in the potential, we have

$$S_{ct} = \int_{D=4} V_{ct} = \int_0^\infty d\rho\, \rho^3 \left(\frac{1}{2} \bar{\phi}^2 \delta_{m^2} + \frac{1}{4} \bar{\phi}^4 \delta_\lambda - FV \right)$$

in $D=4$, for our potential : $\delta_{m^2} = -\frac{3\lambda^2 v^2}{(4\pi)^2 2\varepsilon}$, $\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2 2\varepsilon}$,

and we use $\bar{\phi} = v(\varphi_0 + \Delta \varphi_1) = v\left(\mathrm{th}\frac{z}{2} - \Delta\right)$

$$S_{ct} = \frac{3\lambda^2}{8(4\pi)^2 \varepsilon} \int_{D=4} 3\left(\bar{\phi}^4 - \phi_{FV}^4\right) - 2v^2\left(\bar{\phi}^2 - \bar{\phi}_{FV}^2\right) \simeq -\frac{3}{16\,\varepsilon\Delta^3}.$$

RUNNING OF THE ACTION

$$\beta_\lambda = \frac{d\lambda}{d\ln\mu} = \frac{9\lambda^2}{(4\pi)^2} \Rightarrow \lambda(\mu) = \overset{\lambda_0}{\overset{\shortparallel}{\lambda(\mu_0)}} + \frac{9\lambda_0^2}{(4\pi)^2} \ln\left(\frac{\mu}{\mu_0}\right)$$

COMBINE : $S_R + S_{ct} = S\left(1 - \frac{9\lambda_0}{(4\pi)^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{\mu_0}\right)\right).$

pole arbitrary μ

IV. FLUCTUATIONS

IV. i.) Low-ℓ multipoles

IV. ii) High-ℓ multipoles

IV. i) Low-ℓ

$$\frac{\det \mathcal{O}}{\det \mathcal{O}_{FV}} = \underbrace{\prod_i \frac{\lambda_i}{\lambda_{iFV}}}_{\text{"DIRICHLET" APPROACH}} \overset{\text{reorganize}}{=} \prod_\ell \left(\frac{\lambda_\ell}{\lambda_{\ell FV}}\right)^{d_\ell}$$

where $\psi_\ell(\rho = 0, \infty) = 0$

Gel'fand - Yaglom: $\mathcal{O}_\ell \psi_\ell = 0$, $\mathcal{O}_\ell^{FV} \psi_\ell^{FV} = 0$,

$$R_\ell = \frac{\psi_\ell}{\psi_{\ell FV}}, \quad R_\ell(0) = 1, \quad \dot{R}_\ell(0) = 0.$$

$$\Downarrow$$

$$\frac{\det \mathcal{O}}{\det \mathcal{O}_{FV}} = \prod_\ell \left(R_\ell(\infty)\right)^{d_\ell}$$

ℓ-dependence only here

where : $\ddot{R}_\ell + 2\left(\frac{\dot{\psi}_{\ell FV}}{\psi_{\ell FV}}\right)\dot{R}_\ell = \left(V^{(2)} - V^{(2)}_{FV}\right) R_\ell$

normalization of $\psi_{\ell FV}$ cancels out

makes it stable when $\rho \to \infty$ $V^{(2)} \to V^{(2)}_{FV}$

R_ℓ solution at low ℓ in TW

- First we solve for $\psi_{\ell FV} \longrightarrow \rho^{\frac{D-1}{2}} \psi_{\ell FV}$,

$$\frac{\psi''_{\ell FV}}{\psi_{\ell FV}} = \frac{\nu^2 - 1/4}{(z+r)^2} + \tilde{V}^{(2)}_{FV}, \quad \nu = \ell + \frac{D}{2} - 1,$$

$$\psi_{\ell FV} \simeq C_{FV}\, e^{\underbrace{\left(1 - \frac{3}{2}\Delta + \left(\frac{\nu^2 - 1/4}{2r_0^2} - \frac{21}{8}\right)\Delta^2\right)}\, z},$$

C_{FV} cancels out

enters into the R_ℓ equation, ℓ-dependence comes in @ Δ^2-level, no z-dependence.

- Multiplicative expansion and $x = e^z \in [e^{-r}, e^{\infty}] \simeq [0, \infty]$

$$\psi_{\ell FV} = \prod_{n \geq 0} \psi_{\ell FV n}^{\Delta^n}, \quad R_\ell = \prod_{n \geq 0} R_{\ell n}^{\Delta^n} = R_{\ell 0} R_{\ell 1}^{\Delta} R_{\ell 2}^{\Delta^2} \cdots$$

expand for small Δ

$$R_\ell = R_{\ell 0}\left(1 + \Delta \ln R_{\ell 1} + \Delta^2 \left(\frac{1}{2}\ln^2 R_{\ell 1} + \ln R_{\ell 2}\right) + \dots\right)$$

- The final results are:

$n=0$: $R_{\ell 0} = \frac{1}{(1+x)^2} \longrightarrow 0$,

$n=1$: $\ln R_{\ell 1} = 3(r + \ln x)$, still $R_\ell = 0$

$n=2$: $\ln R_{\ell 2} = \frac{3}{4}\frac{(\ell-1)(\ell+D-1)}{(D-1)^2} x^2$

this beats the $\frac{1}{x^2}$ at $n=0$.

Low-ℓ : $R_e \simeq \Delta^2 \, \ell^{D-1} \, \frac{3}{4} \, \frac{(\ell-1)(\ell+D-1)}{(D-1)^2}$.

* starts at Δ^2, as expected,
* contains a non-trivial ℓ^{D-1} correction from $n=1$ (checked numerically, not there in hist. Dirichlet),
* gives us a negative value at $\ell=0$,
* gives us D zeroes when $\ell=1$, translations,
* does not go to 1 when $\ell \gg 1$ $R_{\ell \gg 1}(\infty) \nrightarrow 1$,

because we took $(\Delta\nu)^2$ to be $\mathcal{O}(\Delta^2)$, thus this is valid only for $\nu < \frac{1}{\Delta}$.

* We can use this R_e for zero removal and get

$$R_1^1 = \frac{\ell^{D-1}}{12} \frac{1}{\lambda \nu^2}.$$

(ask me later how to derive this, if you are interested, does not affect the leading order in Δ^{D-1}, but it gives our Γ the right dimension).

IV. ii) High-ℓ, or generic, multipoles

- We will count $\Delta\nu = \mathcal{O}(1)$, not Δ^1, so we include the multipole dependence already at $n=0$, not $n=2$.

- We now have $\Psi_{\nu FV}$ and R_ν:

$$\Psi_{\nu FV} \simeq e^{k_\nu z} \quad , \quad k_\nu^2 = 1 + \left(\frac{\Delta\nu}{r_0}\right)^2,$$

This enters directly into the eq. for R_ν

$$R_\nu'' + (2k_\nu + 1)\frac{1}{x}R_\nu' = \frac{1}{x^2}\left(\tilde{V}^{(2)} - \tilde{V}^{(2)}_{FV}\right)R_\nu$$

The solution for R_ν is easy to get at $n=0$,

$$n=0 \quad : \quad R_{\nu 0}(\infty) = \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)}$$

$\Delta \to 0$, $k_\nu \to 1$: $\to 0$

$\ell \gg 1$, $k_\nu \gg 1$: $\to 1$ $\boxed{\checkmark}$

but hard to get the Δ^0 corrections

$$U = 3r_0\left(k_\nu - \sqrt{k_\nu^2 - 1}\right)$$

$$\boxed{R_\nu = R_{\nu 0}\, e^{u}}$$

final result for generic ν!

$$\nu = l + \frac{D}{2} - 1, \quad r_0 = \frac{D-1}{3},$$

$$k_\nu^2 = 1 + \left(\frac{\Delta\nu}{r_0}\right)^2$$

Let us summarize and comment on this result:

$$\ln R_\nu = \ln \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1}\right).$$

* l-dependence enters at $n=0$. $k_\nu \sim 1$ reproduces the $R_e \to 0$ result, but: we only included φ_0 and φ_1, no φ_2 or other Δ^2 corrections. These are subleading and won't enter in the final expression.

* The $\mathcal{U}$-factor $\mathcal{U}(k_\nu \to 1) = 3r_0 = D-1$ reproduces the low-l expression e^{D-1}.

* $e^{\mathcal{U}}$ does not affect the leading order of $R_\nu \xrightarrow{k_\nu \gg 1} 1$ but it will give sub-leading $\frac{1}{k_\nu}$ terms, which are crucial for renormalization. Without these, the $\frac{1}{\varepsilon}$ poles and μ-dependence would not cancel.

V.) RENORMALIZED DETERMINANT

V. i) Renormalized rates

V. ii) Finite sum in $D=3$

V. iii) Finite sum in $D=4$

V. iv) Generic Ds

V. i) RENORMALIZED RATES

To compute the determinant, we need to multiply all R_ℓ in Γ, or sum over ℓ in $\ln \Gamma$. This quantity is divergent, but we can regulate the divergence, translate it into $\frac{1}{\varepsilon}$ & $\ln \mu$ parts within a consistent scheme. Approaches are:

1) EFFECTIVE ACTION / FEYNMAN DIAGRAMS,

2) WKB / ζ - FUNCTION FORMALISM.

• Here, we will adopt the ζ-function. Let's see how it works. First, the infinities in $l \gg 1$

$$\ln\left(\frac{\det \sigma}{\det O_{FV}}\right) = \sum_{\nu=\frac{D}{2}-1}^{\infty} d_\nu \ln R_\nu$$

$\nu \gg 1$: $\ln\left(1 + \frac{C_{-1}}{\nu} + \frac{C_{-3}}{\nu^3} + \dots\right)$

large ν, exact for $D = 2,3,4$.

degeneracy factor

$$d_\nu = \frac{2\nu\,(\nu + \frac{D}{2} - 2)!}{(D-2)!\,(\nu - \frac{D}{2} + 1)!} \simeq \frac{2}{(D-2)!}\,\nu^{D-2}.$$

* at large ν : $D \geq 4$

$$\sum_{\nu \gg 1} d_\nu \ln R_{\nu \gg 1} \sim -\frac{3 r_0 (2 - r_0)}{(D-2)!\,\Delta} \sum_\nu \nu^{D-2}\left(\frac{1}{\nu} - \frac{1}{\nu^3} \times \left(\frac{r_0}{2\Delta}\right)^2\right)$$

from $\frac{1}{\nu} \Rightarrow \sum_\nu \frac{\nu^2}{\nu} \sim \nu^2$ divergence,

$\frac{1}{\nu^3} \Rightarrow \sum \frac{\nu^2}{\nu^3} \sim \ln \nu$ divergence.

* The renormalization prescription has to remove those and make the sum finite. The removed part is then added back in the Γ_R. Let's see how this works in $D=3$.

V. ii) FINITE SUM IN D = 3

* Renormalized det :

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{FV}}\right) = \sum_\nu d_\nu \Big(\underbrace{\ln R_\nu - \frac{I_1}{2\nu}}_{\ln R_\nu^a}\Big)$$

[DUNNE & KIRSTEN, DUNNE & MIN]

* The I_1 is derived from WKB / ξ and is given by

$$I_1 = \int_0^\infty d\rho\, \rho \left(V^{(2)} - V^{(2)}_{FV}\right) \simeq -3(2-r_0)\left(\frac{r_0}{\Delta}\right).$$

When multiplied by $\left(-\frac{d_\nu}{2\nu}\right)$, we see that it precisely removes the $\sum \frac{\nu^{D-2}}{\nu} = \sum_\nu =$ lin. divergence.

* There are no $\frac{1}{\varepsilon}$ poles or μ in odd D.

* Now we just have to sum up, using Euler-Maclaurin

$$y = \frac{\Delta \nu}{r_0}, \quad r_0 = \frac{2}{3}, \quad k_\nu = \sqrt{1+y^2}, \quad \ln R_\nu^a = -\frac{2}{y}$$

$$\Sigma_3^f \simeq 2\left(\frac{r_0}{\Delta}\right)^2 \int_{y_0}^\infty dy\, y \left(\ln R_\nu + \frac{2}{y}\right) = \frac{1}{\Delta^2}\,\frac{20+9\ln 3}{27}.$$

* THIS IS IT! The same expression as the one we stated in the intro/summary.

* Let us comment a bit on this result.

* The $\int_{y_0}^{\infty} \cdots dy$ does not depend on the lower boundary $y_0 = \frac{\Delta \nu_0}{r_0}$. We could send it to zero and thus neglect / ignore the low multipoles. This justifies why we "dropped" the low-ℓ part R_ℓ. We only need R_ℓ when $\ell \sim \frac{1}{\Delta}$, which is precisely where R_ν is valid.

* The EuMac procedure / approx. also has corrections:

$$\Sigma_D = \sum_{\nu=\nu_0}^{\infty} \sigma_D = \sum_{\nu=\nu_0}^{\infty} d_\nu \left(\ln R_\nu - \ln R_\nu^a \right)$$

$\nu_0 = \frac{D}{2}-1$, i.e. $\ell = 0$

$$\simeq \underbrace{\Sigma_D^S}_{= \int_{\nu_0}^{\infty} d\nu\, \sigma_D} + \underbrace{\Sigma_D^{bud}}_{= \frac{1}{2}\sigma_D(\nu_0)} + \underbrace{R_p}_{= \text{remainder}} - \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2j}}{(2j)!} \sigma_D^{(2j-1)}(\nu_0).$$

* In $D=3$, $\sigma_3 \simeq \frac{8}{3\Delta}$, which is $< \mathcal{O}(\frac{1}{\Delta^2})$ from the Σ_3^S, so we can safely drop it.

⁂ Thermal Field Theory [Laine, Vourinen, ...]

$$\frac{\Gamma}{V} = \left(\frac{\lambda_-}{2\pi}\right)\left(\frac{S_3}{2\pi T}\right)^{3/2} \sqrt{\frac{\det \mathcal{O}_{FV}}{\det' \mathcal{O}}}\, e^{-S_3/T} .$$

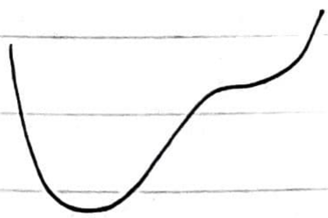
Here : $$S_3 = \underbrace{\frac{1}{\Delta^2}\,\frac{2^5 \pi v}{3^4 \sqrt{\lambda}}}\left(1 - \underbrace{\left(\frac{9\pi^2}{4} - 1\right)\Delta^2}_{\text{NEW!}}\right).$$

$[v] = 1$, so $\frac{S_3}{T}$ is dimensionless

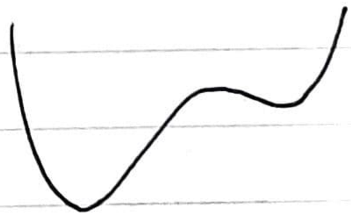
AND : $$\ln \sqrt{\frac{\det \mathcal{O}_{FV}}{\det' \mathcal{O}}} \overset{TW}{\simeq} -\frac{1}{\lambda}\Sigma_3 \ldots$$ nucleation T, P(bubble) ~ 1

* a good approximation when $T_N \sim T_c$

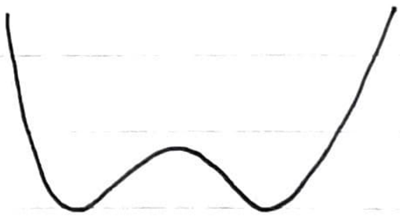
HIGH T



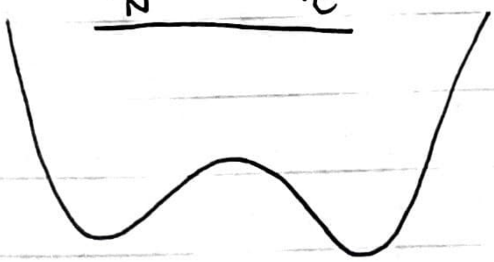
$T \gtrsim T_c$



CRITICAL T

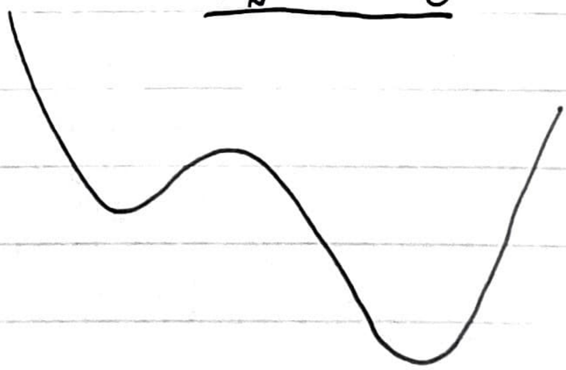


$T_N \lesssim T_c$



TW applies HERE!

$T_N << T_c$



SUPER-COOLED TRANSITIONS

V.iii) FINITE SUM IN D=4

* Again we start with the renormalized ln det calculated in WKB/ζ [DK, DH,...]

$$\ln\left(\frac{\det\sigma}{\det\sigma_{FV}}\right) = \sum_\nu d_\nu \Big(\ln R_\nu \underbrace{- \overset{\text{quadratic div.}}{\frac{1}{2\nu} I_1} + \overset{\text{ln div.}}{\frac{1}{8\nu^3} I_2}}_{\substack{\ln R_\nu^a \\ \vdots \\ \text{asymptotic, subtraction part}}}\Big) \underbrace{- \frac{1}{8}\tilde{I}_2}_{\text{renormalized part}}$$

* In $D=4$, $d_\nu = \nu^2$ and I_1 is the same as before.

$$I_2 = \int_0^\infty d\rho\, \rho^3 \left(V^{(2)2} - V^{(2)2}_{FV}\right) \overset{TW}{=} -3(2-r_0)\left(\frac{r_0}{\Delta}\right)^3, \quad r_0=1$$

$$\tilde{I}_2 = \int d\rho\, \rho^3 \left(V^{(2)2} - V^{(2)2}_{FV}\right)\Big(\underbrace{\frac{1}{\varepsilon}}_{\text{POLES}} + \underbrace{\gamma_E + 1}_{\text{FINITE PART}} + \underbrace{\ln\frac{\mu\rho}{2}}_{\ln\mu\text{ PART}}\Big)$$

$$\tilde{I}_2 \overset{TW}{=} I_2\left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln\left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta}\right)\right).$$

* We have $\tilde{I}_2$, now let's do the finite sum.

* Again : $\Sigma_4 = \Sigma_4^s + \Sigma_4^{bnd}$

$$y = \frac{\Delta V}{r_0} \quad , \quad \ln R_r^a = -\frac{3}{2y} + \frac{3}{8y^3} \, ,$$

$$\Sigma_4^s = \frac{1}{\Delta^3} \int_{y_0}^{\infty} dy \, y^2 \left(\ln R_V + \frac{3}{2y} - \frac{3}{8y^3} \right) \qquad \bigstar$$

$$\overset{TW}{=} \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} + \ln 2y_0 \right).$$

* The TW limit gives us $\frac{1}{\Delta^3}$, just like the action, so Δ^{-D+1}. The λ is gone, it's a λ correction of the $\frac{1}{\lambda}$ piece, which cancels out.

* There is a residual y_0 dependence. This is an artefact of the ζ scheme, FD don't have it. It comes only from the last term in $\bigstar$, that integrates into $\ln y$. It will be cancelled by the "bnd" part of the Eu Mac approximation.

* Let us focus on Σ_4^{bnd}, the boundary terms of the EuMac approximation.

Here : $\nu_0 = \mathcal{O}(1)$ & we have $\Delta\nu_0 = \mathcal{O}(\Delta)$, so we can expand in small Δ

$$\sigma_4^{(j)}(\nu_0) = \frac{3}{8}\,\frac{(-1)^{j+1}\, j!}{\Delta^3\, \nu_0^{\,j+1}} . \qquad \star$$

$\nu_0 \sim \mathcal{O}(1)$ and $j!$ takes over $\Rightarrow$ diverges.

* Solution : split the sum in two pieces

$$\Sigma_4 = \Sigma_4^{low} + \Sigma_4^{high} = \sum_{\nu=\nu_0}^{\nu_1 \sim \frac{1}{\Delta}} \sigma_4 + \sum_{\nu=\nu_1+1}^{\infty} \sigma_4 .$$

* The high part is $\sim \Sigma_4^{S}$ and $\star$ vanishes.

* The low part

$$\Sigma_4^{low} = -\frac{3}{8\Delta^3} \sum_{\underset{\nu_0 \text{ in } D=4}{\underset{\shortparallel}{\nu=1}}}^{\nu_1} \frac{1}{\nu} = -\frac{3}{8\Delta^3} H(\nu_1) \simeq -\frac{3}{8\Delta^3}\left(\ln \nu_1 + \gamma_E\right)$$

$$\Sigma_4^{high} = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} + \ln 2\underset{\Delta\nu_1}{\underset{\shortparallel}{y_1}}\right)$$

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ν_1 cancels out

* The dependence on the (arbitrary) point of separation at ν_1 is reassuringly gone.

$$\Sigma_4 = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta\right)$$

RENORMALIZED RATE

$$\ln\left(\frac{\det O}{\det O_{FV}}\right) = \Sigma_4 - \frac{1}{8}\tilde{I}_2$$ & finally:

$$\ln\frac{\Gamma}{V} \ni -\left(S_R + S_{ct} + \frac{1}{2}\left(\Sigma_4 - \frac{1}{8}\tilde{I}_2\right)\right)$$

$$= -S - \frac{1}{\Delta^3}\,\frac{45-4\pi\sqrt{3}}{192}.$$

* no μ-dependence, no $\frac{1}{\varepsilon}$ poles,
* μ_0 also canceled out with $\sqrt{\lambda}v$ in $\tilde{I}_2$,
* the $\gamma_E + \ln 2\Delta$ from Σ_4 cancels with $\tilde{I}_2$.

$\boxed{\text{DONE!}}$ This is what was quoted in the summary.

V. iv) Generic Ds

- The $\ln R_v^a$ in the ξ approach was defined by successive subtractions in $\frac{1}{r}, \frac{1}{r^3}, \frac{1}{r^5}, \ldots$ etc.
- The general formula for Γ/V in any D is not known (to me). However, we do know the minimal subtraction $\ln R_v^a$ from expanding $\ln R_v$ in large r and extracting the $\frac{1}{r^p}$ terms.
- These asymptotic terms are given by $\left(y = \frac{\Delta V}{r_0}\right)$

$$\ln R_{v_0}^a = -2 \sum_{j=1}^{m} \frac{1+2^{-2j+1}}{2j-1} \sum_{p=0}^{m-j} \binom{-j+1/2}{p} y^{-2(j+p)+1},$$

$$U^a = -3r_0 \sum_{j=1}^{m} \binom{1/2}{j} (-1)^j y^{-2j+1} \sum_{p=0}^{m-j} \binom{-j+1/2}{p} y^{-2p}.$$

where $D = 2m$ for even and $D = 2m+1$ for odd.

* Now we simply repeat the EnMac steps.

* It is slightly easier to do it for odd Ds :

$$\Sigma_{2m+1} = \frac{2}{(D-2)!}\left(\frac{v_0}{\Delta}\right)^{D-1} \frac{(-1)^{m+1}}{m\binom{m-1/2}{m}} \left(\frac{1}{m} + {}_2F_1\left(1,\frac{1}{2}-m;\frac{3}{2};\frac{1}{4}\right) - \frac{3}{2}v_0\right)$$

$$= \frac{2}{(D-2)!}\left(\frac{v_0}{\Delta}\right)^{D-1} \begin{cases} \frac{5}{6} + \frac{3}{8}\ln 3 \quad , & D=3 \\ -\frac{37}{240} - \frac{9}{64}\ln 3 \quad , & D=5 \\ -\frac{3}{224} + \frac{9}{128}\ln 3 \quad , & D=7, \ \ldots \end{cases}$$

* The even D's are a bit more complicated. We get the dependence on $\ln v_1$ again, by splitting the sum. However, it cancels out between the low and high ℓ parts :

$$\Sigma_D^{low} = \frac{2}{(D-2)!}\left(\frac{v_0}{\Delta}\right)^{D-1} \begin{cases} 5/2\,(\gamma_E + \ln v_1), & D=2 \\ -3/8\,(\gamma_E + \ln v_1), & 4 \\ \frac{1}{10}\,(\gamma_E - \zeta(3) + \ln v_1, & 6, \ldots \end{cases}$$

$$\Sigma_D^{high} = \frac{2}{(D-2)!}\left(\frac{v_0}{\Delta}\right)^{D-1} \begin{cases} \frac{\pi}{2\sqrt{3}} - \frac{11}{4} - \frac{5}{2}\ln 2y_1, & 2 \\ -\frac{\pi}{8\sqrt{3}} + \frac{3}{32} + \frac{3}{8}\ln 2y_1, & 4 \\ \frac{3\sqrt{3}\pi}{160} - \frac{211}{4800} - \frac{1}{10}\ln 2y_1, & 6 \end{cases}$$

VI. SUMMARY & OUTLOOK

* We solved the 45-year old problem, "posed" or "left" by Callan & Coleman and found an analytic closed form solution for:

- the bounce action up to $\mathcal{O}(\Delta^4)$,
- the functional determinant,
- explicit $\frac{\Gamma}{V}$ for $D=3,4$ in $\overline{MS}$.

* We also derived generic expression for the finite $\ln \det \mathcal{O}$ for <u>any</u> D.

* <u>Outlook</u>: a) $D\varphi \to D\varphi_\perp, D\psi, DA$,

b) $\overline{\varphi} \to \varphi_{PB}$, $D\varphi$ analytically,

c) Pheno = PhTrs, GWs, baryogenesis, ...

Appendix : Zero Removal

* At $l=1$ we have D degenerate zeroes that are multiplied with other non-zero λ's with $l=1$.

* We remove them perturbatively by off-setting $V^{(2)}$ inside $\mathcal{O}_{l=1}$ by a small dimensional μ_ε^2

$$\mathcal{O}_1 \psi_1 = 0 \quad , \quad \psi_1(\infty) = 0 \implies (\mathcal{O}_1 + \mu_\varepsilon^2)\, \psi_1^\varepsilon = 0.$$

* We can solve for ψ_1^ε by $V^{(2)} \to V^{(2)} + \Delta^2 \mu_\varepsilon^2$, because the l-dependence and the $l=1$ zero starts at Δ^2.

Then :
$$\ln \underbrace{R^\varepsilon_{12}}_{l=2 \;\; \mathcal{O}(\Delta^2)} = \frac{1}{12} \frac{\mu_\varepsilon^2}{\lambda v^2} X^2$$

&
$$R_1'(\infty) = \underbrace{R_{l0}(\infty)}_{\simeq \frac{1}{X^2}}\, l^{D-1} \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\Delta^2 \mu_\varepsilon^2} \Delta^2 \ln R^\varepsilon_{21} = \frac{l^{D-1}}{12} \times \frac{1}{\lambda v^2}.$$

Why? Because:
$$R_1^\varepsilon(\infty) = \frac{(\mu_\varepsilon^2 + \overset{\nearrow 0}{\gamma_1}) \prod_{n=2}^{\infty} \gamma_{\vec{n}}}{\prod_{n=1}^{\infty} \gamma_{\vec{n}}^{FV}} = \frac{\psi_1^\varepsilon(\infty)}{\psi_1^{FV}(\infty)} = \mu_\varepsilon^2 R_1'(\infty)$$