Feynman diagramatic approach for functional determinants

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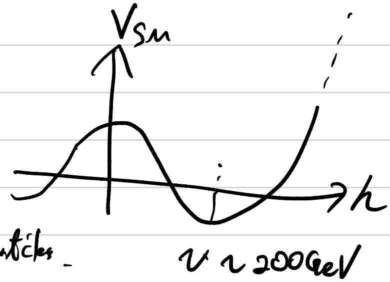
Contents

1. Introduction

Vacuum Stability in SM

$$V_{SM}^{tree} = -\mu_H^2 |H|^2 + \lambda |H|^4$$

$$\ni -\frac{\mu_H^2}{2} h^2 + \frac{\lambda}{4} h^4$$



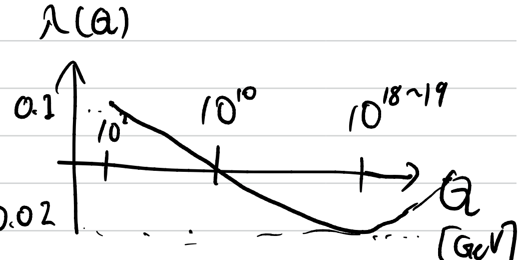
mass of particles

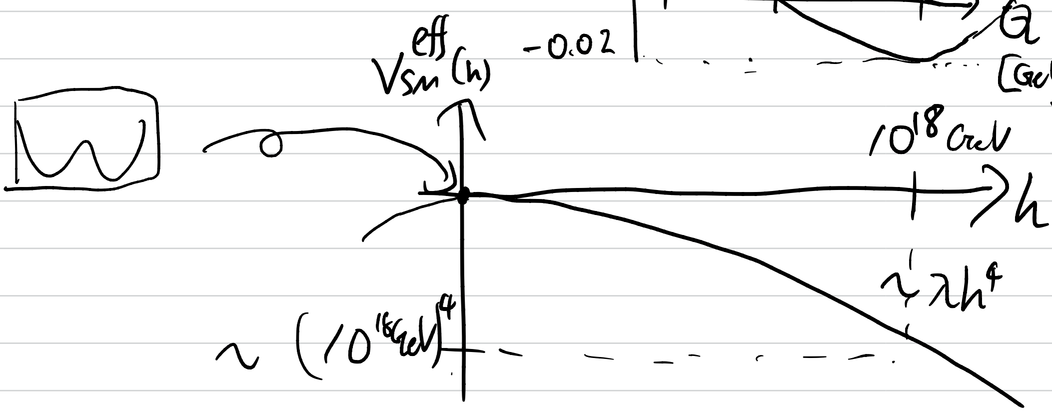
$$V_{SM}^{eff} \ni \bigcirc \ln \frac{M^2(h)}{Q^2} + \bigcirc \left[\ln \frac{M^2(h)}{Q^2} \right]^2 + \cdots$$

renormalization scale.

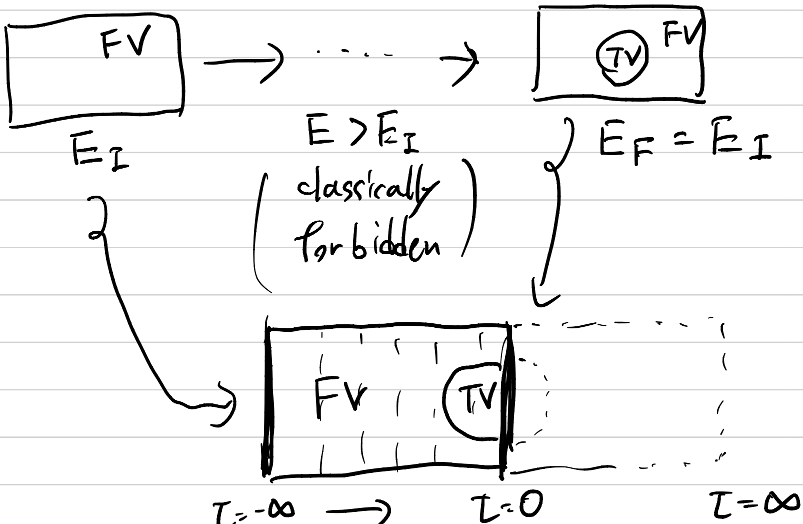
$h \gg v$, resummation by renormalization group evolution

$$\to V_{SM}^{eff} \simeq \frac{\lambda(Q \sim h)}{4} h^4$$





Bubble nucleation rate



Bounce ϕ_B

O(4) symmetric.

$\phi_B = \phi_B(r)$, $\partial_r^2 \phi + \frac{3}{r}\partial_r \phi = \frac{dV}{d\phi}$

[Callan, Coleman, '77]

$$\gamma = A e^{-B}$$

$$B = S[\phi_B] - S[\phi_{FV}]$$

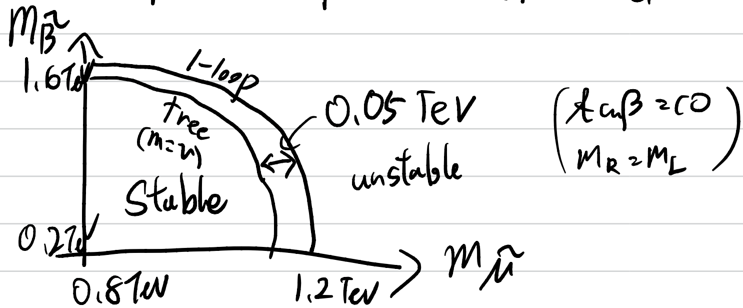
zero mode subtraction

$$A = \frac{B}{4\pi^2}\left[\frac{\mathrm{Det}' S''|_{\phi_B}}{\mathrm{Det}\, S''|_{\phi_F}}\right]^{-1/2}$$

$$A \Rightarrow \begin{cases} \text{dimensional analysis} = m^4 \\ \text{or} \\ \text{explicit calculation} \\ \quad = m^4 e^{-B - \delta B^{1\text{-loop}}} \end{cases}$$

$$\delta S^{1\text{-loop}} \sim \mathcal{O}(10\%) \times \underset{\substack{\vee \\ 400}}{B} \sim \mathcal{O}(10\text{-}100)$$

ex) MSSM parameter space that explains $g_\mu - 2$



[S. Chigusa, T. Moroi, YS, '22]

$\Rightarrow$ The effect of A is not negligible!

2. Functional determinant

$$\mathcal{M}_{IJ} = \frac{\partial^2 S}{\partial \phi_I \partial \phi_J}[\phi_B] \quad , \quad \hat{\mathcal{M}}_{IJ} = \frac{\partial^2 S}{\partial \phi_I \partial \phi_J}[\phi_F]$$

$$\frac{\mathrm{Det}\,\mathcal{M}}{\mathrm{Det}\,\hat{\mathcal{M}}} = \prod_{\ell=0}^{\infty} \left(\frac{\mathrm{Det}\,\mathcal{M}_\ell}{\mathrm{Det}\,\mathcal{M}_\ell} \right)^{(\ell+1)^2}$$

MSSM $\sim 10\times 10$

$$\mathcal{M}_\ell = \begin{pmatrix} -\partial_r^2 - \frac{3}{r}\partial_r + \frac{L^2}{r^2} + \bullet\,\phi_B^2 & \cdots \\ \bullet\,\partial_r + \bullet\,(\partial_r \phi_B) & \ddots \\ \vdots & \end{pmatrix}$$

$n \times n$ matrix

(Generalized) Gelfand-Yaglom theorem

(※ Not for general $\mathcal{M}_\ell$)

$$\frac{\mathrm{Det}\,\mathcal{M}_\ell}{\mathrm{Det}\,\hat{\mathcal{M}}_\ell} = \left(\frac{\det \Psi_\ell(\infty)}{\det \hat{\Psi}_\ell(\infty)} \right) \Big/ \left(\frac{\det \Psi_\ell(0)}{\det \hat{\Psi}_\ell(0)} \right)$$

$$\overset{(\wedge)}{\Psi}_\ell(r) = \left(\overset{(\wedge)}{\psi}{}_\ell^1(r), \cdots, \overset{(\wedge)}{\psi}{}_\ell^n(r) \right)$$

$\overset{(\wedge)}{\mathcal{M}}_\ell \overset{(\wedge)}{\psi}{}_\ell^i = 0$, $\overset{(\wedge)}{\psi}{}_\ell^i$: independent solutions regular at $r=0$

$$\frac{\mathrm{Det}\,\mathcal{M}}{\mathrm{Det}\,\hat{\mathcal{M}}} = \prod_{\ell=0}^{\infty} \Bigg(\underbrace{\frac{\mathrm{Det}\,\mathcal{M}_\ell}{\mathrm{Det}\,\hat{\mathcal{M}}_\ell}}_{\text{finite}} \Bigg)^{(\ell+1)^2} \to \text{diverge.}$$

Regularization / Renormalization

$\ln \mathrm{Det}(1+\hat{M}^{-1}\delta M)$
$\mathrm{Tr}\ln(1+\hat{M}$

$$
\begin{aligned}
\ln \mathrm{Det}\, M &= \ln \mathrm{Det}(\hat{M}+\delta M) \\
&= \ln \mathrm{Det}\,\hat{M} + \mathrm{Tr}\ln(1+\hat{M}^{-1}\delta M) \\
&= \ln \mathrm{Det}\,\hat{M} + \mathrm{Tr}\,\hat{M}^{-1}\delta M - \frac{1}{2}\mathrm{Tr}(\hat{M}^{-1}\delta M)^2 \\
&\quad + \frac{1}{3}\mathrm{Tr}(\hat{M}^{-1}\delta M)^3 - \cdots\cdots \\
&= \Bigg(\ln \mathrm{Det}\,\hat{M} + \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\, \underset{\frac{1}{k^2+m^2}}{\Delta(k)}\, \widetilde{\delta M}(0) \\
&\quad - \frac{1}{2}\mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\, \Delta(k)\,\widetilde{\delta M}(k-p)\, \Delta(p)\,\widetilde{\delta M}(p-k) \Bigg) \\
&\quad + \text{finite}
\end{aligned}
$$

finite

$$
\left[\ln\frac{\mathrm{Det}\,M}{\mathrm{Det}\,\hat{M}}\right]_{\overline{MS}} = \left[\ln\frac{\mathrm{Det}\,M}{\mathrm{Det}\,\hat{M}} - \mathrm{Tr}\,\hat{M}^{-1}\delta M + \frac{1}{2}\mathrm{Tr}(\hat{M}^{-1}\delta M)^2\right]
$$

$$
+ \left[\mathrm{Tr}\,\hat{M}^{-1}\delta M - \frac{1}{2}\mathrm{Tr}(\hat{M}^{-1}\delta M)^2\right]_{\overline{MS}}
$$

Dimensional regularization + $\overline{MS}$ renormalization

Finite part

$$\ln \frac{\mathrm{Det}\, M}{\mathrm{Det}\, \hat{M}} = \sum_{\ell=0}^{\infty} (\ell+1)^2 \ln \frac{\det \Phi_\ell}{\det \hat{\Phi}_\ell}$$

$$(\hat{M} + \delta M)\Phi_\ell = 0$$

$$\Rightarrow \hat{M} \Phi_\ell^{(n+1)} = -\delta M \Phi_\ell^{(n)}, \quad \Phi_\ell^{(0)} = \hat{\Phi}_\ell, \quad \Phi_\ell^{(n)}(0) = 0$$

$$\ln \frac{\mathrm{Det}\, M}{\mathrm{Det}\, \hat{M}} = \sum_{\ell=0}^{\infty} (\ell+1)^2 \Big[\underbrace{\mathrm{tr}\, \hat{\Phi}_\ell^{-1} \Phi_\ell^{(1)}}_{O(\delta M)} + \underbrace{\mathrm{tr}\, \hat{\Phi}_\ell^{-1} \Phi_\ell^{(2)} - \frac{1}{2} \mathrm{Tr}(\hat{\Phi}_\ell^{-1} \Phi_\ell^{(1)})^2}_{O(\delta M^2)} + O(\delta M^3) \Big]$$

$$\left[\ln \frac{\mathrm{Det}\, M}{\mathrm{Det}\, \hat{M}} - \mathrm{Tr}\, M^{-1} \delta M + \frac{1}{2} \mathrm{Tr}(M^{-1}\delta M)^2 \right]$$

$$= \sum_{\ell=0}^{\infty} (\ell+1)^2 \Big[\ln \frac{\det \Phi}{\det \hat{\Phi}} - \mathrm{tr}\, \hat{\Phi}_\ell^{-1} \Phi_\ell^{(1)} - \mathrm{tr}\, \hat{\Phi}_\ell^{-1} \Phi_\ell^{(2)} + \frac{1}{2} \mathrm{tr}(\hat{\Phi}_\ell^{-1} \Phi_\ell^{(1)})^2 \Big]$$

Divergent part

$$I_1 \equiv \mathrm{Tr}\, \hat{M}^{-1} \delta M = \mathrm{tr}\, \widetilde{\delta M}(0) \underbrace{\int \frac{d^4k}{(2\pi)^4} \Delta(k)}_{\sim G(0)}$$

$$= \mathrm{tr}\, G(0) \int d^4x\, \delta M(x)$$

$$\widetilde{\delta M}(k) = \int d^4x\, e^{-ikx} \delta M(x), \quad G(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \Delta(k)$$

$$I_2 \equiv \mathrm{Tr}\, \hat{M}^{-1} \delta M\, \hat{M}^{-1} \delta M$$

$$= \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \Delta(k)\, \widetilde{\delta M}(k-p) \times \Delta(p)\, \widetilde{\delta M}(p-k)$$

$$= \mathrm{tr} \int d^4x \int d^4y\, G(y-x)\, \delta M(x) \times G(x-y)\, \delta M(y)$$

$$X = \frac{x+y}{2}, \quad Y = x - y$$

$$= \mathrm{tr} \int d^4X d^4Y \, G(-Y)\, \delta M\left(X + \frac{Y}{2}\right)$$
$$\times G(Y)\, \delta M\left(X - \frac{Y}{2}\right)$$

$$G(Y) \sim \frac{K_1(m|Y|)}{|Y|} \sim \frac{1}{|Y|^2}$$

$$G(Y) = e^{-\frac{\Lambda^2}{2}|Y|^2} G(Y) + \left(1 - e^{-\frac{\Lambda^2}{2}|Y|^2}\right) G(Y)$$

$$I_{2,div} \equiv \mathrm{tr} \int d^4X d^4Y e^{-\Lambda^2|Y|^2} G(-Y)\, \delta M(X)$$
$$\times G(Y)\, \delta M(X)$$

$$= C^{IJKL} \int d X \, \delta M_{JK}(X)\, \delta M_{LI}(X)$$

$$C^{IJKL} \equiv \int d^4Y \, G_{IJ}(-Y)\, G_{KL}(Y)\, e^{-\Lambda^2|Y|^2}$$
$$= \int d^4Y \int \frac{d^4k}{(2\pi)^4} \Delta_{IJ}(k) e^{-ikY} \int \frac{d^4p}{(2\pi)^4} \Delta_{KL}(p)$$
$$\times e^{ipY} e^{-\Lambda^2|Y|^2}$$

$$
= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \Delta_{IJ}(k) \Delta_{KL}(p) \times \int d^4r \, e^{-\Lambda^2 |r|^2 - i(k-p)\cdot r}
$$

$$
= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \left(\frac{\sqrt{\pi}}{\Lambda}\right)^4 \Delta_{IJ}(k) \Delta_{KL}(p) \times e^{-\frac{1}{4\Lambda^2}(k-p)^2}
$$

$$
= \int \frac{d^4k}{(2\pi)^4} \left(\frac{\sqrt{\pi}}{\Lambda}\right)^4 e^{-\frac{k^2}{4\Lambda^2}} \left[\int \frac{d^4p}{(2\pi)^4} \Delta_{IJ}(p+k) \Delta_{KL}(p) \right]
$$

$$
= \frac{\pi^2}{\Lambda^4} \int \frac{d^4k}{(2\pi)^4} e^{-\frac{k^2}{4\Lambda^2}} \left[\int \frac{d^4p}{(2\pi)^4} \Delta_{IJ}(p+k) \Delta_{KL}(p) \right]_{AS}
$$

3. Standard Model and beyond.

$$\phi_B(r) = \sqrt{-\frac{8}{\lambda}}\,\frac{R}{R^2+r^2}$$, R : arbitrary at tree level.

<u>Physical Higgs fluctuation</u>

$$M = -\partial^2 + 3\lambda\phi_B^2(|x|)$$

$$\hat{M} = -\partial^2 \quad , \quad \delta M = 3\lambda\phi_B^2$$

$$\Delta(k) = \frac{1}{k^2}$$

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot x}$$

$$= \frac{1}{(2\pi)^4}\int_0^\pi d\theta \int_0^\pi d\varphi \int_0^{2\pi} d\eta\, \sin^2\theta \sin\varphi \int dk\, k\, e^{ikx}$$

$$= \frac{1}{(2\pi)^2}\frac{1}{x^2}$$

$$G(0) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} \equiv 0 \quad (\overline{MS}\text{ prescription})$$

$$I_1 = 0$$

$$G^D(x) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} e^{ikx}$$

$$= \Omega_{D-2} \frac{1}{(2\pi)^D} \int_0^\pi d\theta \sin^{D-2}\theta \int dk\, k^{D-1}$$

$$\times \frac{1}{k^2} e^{ikx\cos\theta}$$

$$= \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \frac{1}{|x|^{D-2}}$$

$2-\varepsilon$

$$I_{2,div} = \int d^4x\, \delta M^2(x) \int d^{4-2\varepsilon} r \left(\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}}\right)^2 \frac{1}{|r|^{4-4\varepsilon}} e^{-\Delta^2 |r|^2}$$

$$= \int d^4x\, \delta M^2(x)\, \Omega_{3-2\varepsilon} \left(\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}}\right)^2 \int dr\, r^{2\varepsilon-1} e^{-\Delta^2 |r|^2}$$

$= ?$ (missing something)

divergent, $I_1 = 0$

I_2 div, $C^{1111} = \frac{\pi^2}{\Lambda^4}\int\frac{d^4k}{(2\pi)^4} e^{-\frac{k^2}{4\Lambda^2}}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\frac{1}{(p+k)^2}$

$$\left(= \frac{\pi^2}{\Lambda^4}\int\frac{d^4k}{(2\pi)^4} e^{-\frac{k^2}{4\Lambda^2}}\left[\frac{1}{\varepsilon}+2-\ln\frac{k^2}{\mu^2}\right]\frac{1}{16\pi^2}\right.$$

$$= \frac{\pi^2}{\Lambda^4}\frac{2\pi^2}{(2\pi)^4}\int dk\, k^3 e^{-\frac{k^2}{4\Lambda^2}}\left[\frac{1}{\varepsilon}+2-\ln\frac{k^2}{\mu^2}\right]\frac{1}{16\pi^2}$$

$\times \int d^4x\, \delta M^2(x)$

$$= \left[\frac{1}{\varepsilon}+1+\gamma-\ln\frac{4\Lambda^2}{\mu^2}\right]\frac{1}{16\pi^2}$$

finite 1,

$$2\int d^4x\, d^4r\, \frac{1}{(2\pi)^4}\frac{1}{|r|^4}\,\delta M\left(x+\frac{r}{2}\right)\delta M\left(x-\frac{r}{2}\right)$$

$$e^{-\frac{\Lambda^2}{2}|r|^2}\left(1-e^{-\frac{\Lambda^2}{2}|r|^2}\right)$$

$$\simeq \int d^4x\, \delta M^2(x)\,\frac{2}{(2\pi)^4}\int d^4r\,\frac{1}{|r|^4}e^{-\frac{\Lambda^2}{2}|r|^2}\left(1-e^{-\frac{\Lambda^2}{2}|r|^2}\right)$$

$$= \int d^4x\, \delta M^2(x)\,\frac{1}{16\pi^2}2\ln 2$$

finite 2,

$$\int d^4x\, d^4y\,\frac{1}{(2\pi)^4}\frac{1}{|x-y|^4}\,\delta M(x)\,\delta M(y)$$

$$\left(1-e^{-\frac{\Lambda^2}{2}|x-y|^2}\right)^2$$

$$\int d^4x \frac{1}{(2\pi)^2} \frac{1}{|x|^2} \left(1 - e^{-\frac{\Lambda^2}{2}|x|^2}\right) e^{-ik\cdot x}$$

$$= \frac{1}{(2\pi)^2} 4\pi \int d\theta \sin^2\theta \int dr\, r^3 \frac{1}{r^2}\left(1 - e^{-\frac{\Lambda^2}{2}r^2}\right) e^{-ikr\cos\theta}$$

$$= \frac{1}{k^2} e^{-\frac{k^2}{2\Lambda^2}}$$

$$\frac{1}{(2\pi)^2} \frac{1}{|x-y|^2} \left(1 - e^{-\frac{\Lambda^2}{2}|x-y|^2}\right)$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{-\frac{k^2}{2\Lambda^2}} e^{ik\cdot(x-y)}$$

$$\int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{1}{k^2} \frac{1}{p^2} e^{-\frac{k^2+p^2}{2\Lambda^2}}$$

$$\times e^{i(k-p)(x-y)} \delta M(x)\, \delta M(y)$$

$$= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{1}{k^2 p^2} e^{-\frac{k^2+p^2}{2\Lambda^2}} \widetilde{\delta M}(k-p)$$

$$\widetilde{\delta M}(p-k)$$

$$k = p + \frac{k}{2}, \quad p = p - \frac{k}{2}$$

$$= \int \frac{d^4 k}{(2\pi)^4} \widetilde{\delta u}^2(k) \, e^{-\frac{k^2}{4\Lambda^2}} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-\frac{p^2}{\Lambda^2}}}{(p+\frac{k}{2})^2 (p-\frac{k}{2})^2}$$

$$\int \frac{d^4 p}{(2\pi)^4} \frac{e^{-\frac{p^2}{\Lambda^2}}}{(p^2 + pk + \frac{k^2}{4})(p^2 - pk + \frac{k^2}{4})}$$

$$= \frac{1}{16\pi^2} \left[1 - \gamma + \ln \Lambda^2 \right]$$

$$= \int \frac{d^4 K}{(2\pi)^4} \widetilde{\delta M}^2(K) \, \frac{1}{16\pi^2} \left[1 - \gamma + \ln \frac{\Lambda^2}{K^2} \right]$$

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \widetilde{\delta M}^2(k)$$

$$\times \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + 1 + \gamma - \ln\frac{4\Lambda^2}{\mu^2} + 2\ln 2 + 1 - \gamma + \ln\frac{\Lambda^2}{k^2}\right]$$

$$= \int \frac{d^4k}{(2\pi)^4} \widetilde{\delta M}^2(k) \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + 2 - \ln\frac{k^2}{\mu^2}\right]$$

$$\left(\begin{aligned}\widetilde{\delta M}(k) &= \int d^4x\, 3\lambda\left(-\frac{8}{\lambda}\right)\left(\frac{R}{R^2+|x|^2}\right)^2 e^{-ikx} \\ &= -3\cdot 8\cdot 4\pi \int d\theta \sin^2\theta \int dr\, r^3 \left(\frac{R}{R^2+r^2}\right)^2 \cdot e^{-ikx\cos\theta} \\ &= -48\pi^2 R^2 K_0(kR)\end{aligned}\right)$$

$$\Big(= 6\left(\frac{1}{\varepsilon}+2\right) + 2\left(-1 + 6\gamma + 6\ln\frac{\mu R}{2}\right)$$

$$= 12\left[\frac{1}{2\varepsilon} + 1 - \frac{1}{6} + \gamma + \ln\frac{\mu R}{2}\right]$$

$$= 12\left[\frac{1}{2\varepsilon} + \frac{5}{6} + \gamma + \ln\frac{\mu R}{2}\right]$$

ELVAS …. 1-loop decay rates for Fubini instanton with arbitrary couplings to fields.

4. Minimal regularization

$\partial^2 |k\rangle = -k^2 |k\rangle$: D-dimensional momentum space

∂

$\partial^2 |l,a,\lambda\rangle = -\lambda^2 |l,a,\lambda\rangle$: D-dimensional angular/radial momentum space

$$\partial^2 |l,a\rangle = \left[\partial_r^2 + \frac{D-1}{r}\partial_r - \frac{l(l+D-2)}{r^2}\right] |l,a\rangle$$

$$\# a = (D+2l-2)\frac{\Gamma(l+D-2)}{\Gamma(l+1)\Gamma(D-1)} \equiv w_l$$

$$\langle r|l,a,\lambda\rangle = \sqrt{\lambda}\, r^{\frac{2-D}{2}} J_\nu(\lambda r)\, |l,a\rangle$$

$$\nu = \frac{D}{2} + l - 1$$

$$\langle l,a,\lambda|l',a',\lambda'\rangle = \delta(\lambda-\lambda')\,\delta_{ll'}\,\delta_{aa'}$$

$$\hat{M} = -\partial^2 + m^2 \quad , \quad \delta M = \delta M(r)$$

$$\mathrm{Tr}\, \hat{M}^{-1} \delta M$$

$$= \sum_{l,a} \int d\lambda \int dr\, r^{D-1} \langle l,a,\lambda| \frac{1}{\lambda^2+m^2} |r\rangle\langle r| \delta M(r) |l,a,\lambda\rangle$$

$$\left(= \sum_l w_l \int d\lambda \frac{\lambda}{\lambda^2+m^2} \int dr\, r\, J_\nu^2(\lambda r)\, \delta M(r) \right.$$

$$= \sum_{\ell} w_\ell \int dr\, r\, I_\nu(mr) K_\nu(mr)\, \delta M(r)$$

$$\mathrm{Tr}\, \hat{M}^{-1} \delta M \hat{M}^{-1} \delta M$$

$$= \sum_{\ell,a} \sum_{\ell',a'} \int d\lambda d\lambda' \int dr\, r^{D-1} \int dr'\, r'^{D-1} \langle \ell, a, \lambda | r \rangle \frac{\delta M(r)}{\lambda^2 + m^2}$$

$$\times \langle r | \ell', a', \lambda' \rangle \langle \ell', a', \lambda' | r' \rangle \frac{\delta M(r')}{\lambda'^2 + m^2} \langle r' | \ell, a, \lambda \rangle$$

$$= \sum_{\ell} w_\ell \int d\lambda d\lambda' \int dr\, r \int dr'\, r'\, \delta M(r)\, \delta M(r')$$

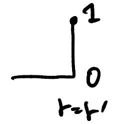
$$\times \frac{\lambda}{\lambda^2 + m^2} \frac{\lambda'}{\lambda'^2 + m^2} J_\nu(\lambda r) J_\nu(\lambda' r) J_\nu(\lambda' r') J_\nu(\lambda r')$$

$$\int_0^\infty d\lambda\, \frac{\lambda}{\lambda^2 + m^2} J_\nu(\lambda r) J_\nu(\lambda r')$$

$$= \begin{cases} I_\nu(mr) K_\nu(mr') & r < r' \\ I_\nu(mr') K_\nu(mr) & r > r' \end{cases}$$

$$= \sum_{\ell} 2 w_\ell \int_0^\infty dr\, r \int_r^\infty dr'\, r'\, \delta M(r)\, \delta M(r')$$

$$\times I_\nu^2(mr) K_\nu^2(mr')$$

$\left(\frac{r}{r'}\right)^\infty$ 

$$0 < mr \ll \sqrt{\nu+1}$$

$$I_\nu(mr) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{mr}{2}\right)^\nu \left[1 + \frac{1}{\nu+1}\left(\frac{mr}{2}\right)^2\right]$$

$$K_\nu(mr) \simeq \frac{\Gamma(\nu)}{2}\left(\frac{2}{mr}\right)^\nu \left[1 - \frac{1}{\nu-1}\left(\frac{mr}{2}\right)^2\right]$$

$$I_\nu(mr)K_\nu(mr') \simeq \frac{\Gamma(\nu)}{2\Gamma(\nu+1)}\left(\frac{r}{r'}\right)^\nu$$

$$= \frac{1}{D+2(\ell-1)}\left(\frac{r}{r'}\right)^\nu$$

$$\sum_\ell w_\ell I_\nu(mr)K_\nu(mr) \ni \sum_\ell \frac{\Gamma(D+\ell-2)}{\Gamma(D-1)\Gamma(\ell+1)}\left[1 - 2\frac{\left(\frac{mr}{2}\right)^2}{\nu^2-1}\right]$$

$$\sum_\ell w_\ell I_\nu^L(mr)K_\nu^2(mr') \ni \sum_\ell \frac{1}{D+2(\ell-1)}\frac{\Gamma(D+\ell-2)}{\Gamma(D-1)\Gamma(\ell+1)}\left(\frac{r}{r'}\right)^\nu$$

$$\int_r^\infty dr' \left(\frac{r}{r'}\right)^{2\nu} r' = \frac{r^2}{2\nu-2} = \frac{r^2}{D+2\ell-4}$$

$D<2$

$=0$

ℓ (but then may be finite terms

$$\mathrm{Tr}\, M^{-1}\delta M \ni \left[\int dr\, r\, \delta M(r)\right] \frac{1}{\Gamma(D-1)} \sum_{\ell=0}^{\infty} \frac{\Gamma(D+\ell-2)}{\Gamma(\ell+1)}$$

$$- m^2 \left[\int dr\, r^3\, \delta M(r)\right] \frac{1}{2\Gamma(D-1)}$$

$$\times \sum_{\ell=1}^{\infty} \frac{\Gamma(D+\ell-2)}{\Gamma(\ell+1)} \frac{1}{\left(\frac{D}{2}+\ell-1\right)^2-1}$$

$$\overset{\boxed{?}}{=} \frac{-m^2}{16\pi^2}\left[2\pi^2 \int dr\, r^3\, \delta M(r)\right]\left[\frac{1}{\varepsilon}+\frac{3}{2}\right]$$

$\ln \frac{m^2}{\mu^2}$?

$$\mathrm{Tr}\, M^{-1}\delta M\, M^{-1}\delta M$$

$$\ni \left[\int dr\, r^3\, \delta M^2(r)\right] \frac{1}{2\Gamma(D-1)}$$

$$\times \sum_{\ell=1}^{\infty} \frac{1}{\frac{D}{2}+\ell-1} \frac{1}{\frac{D}{2}+\ell-2} \frac{\Gamma(D+\ell-2)}{\Gamma(\ell+1)}$$

$$\overset{\boxed{?}}{=} \frac{1}{16\pi^2}\left[2\pi^2 \int dr\, r^3\, \delta M^2(r)\right]\left(\frac{1}{\varepsilon}+3\right)$$

→ need to check against SM