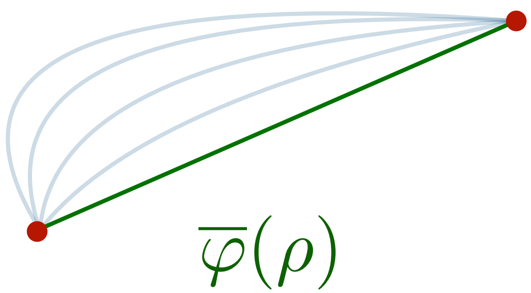
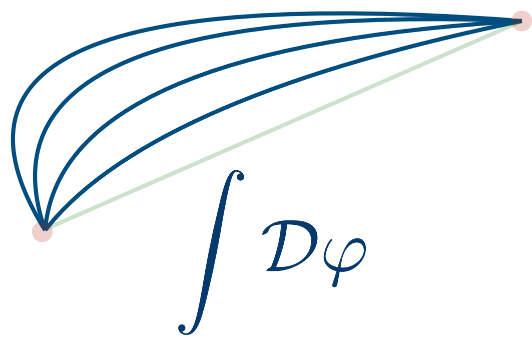
Univerza *v Ljubljani*
Fakulteta za *matematiko in fiziko*

IJS

Polygonal bounces, FindBounce and prefactors



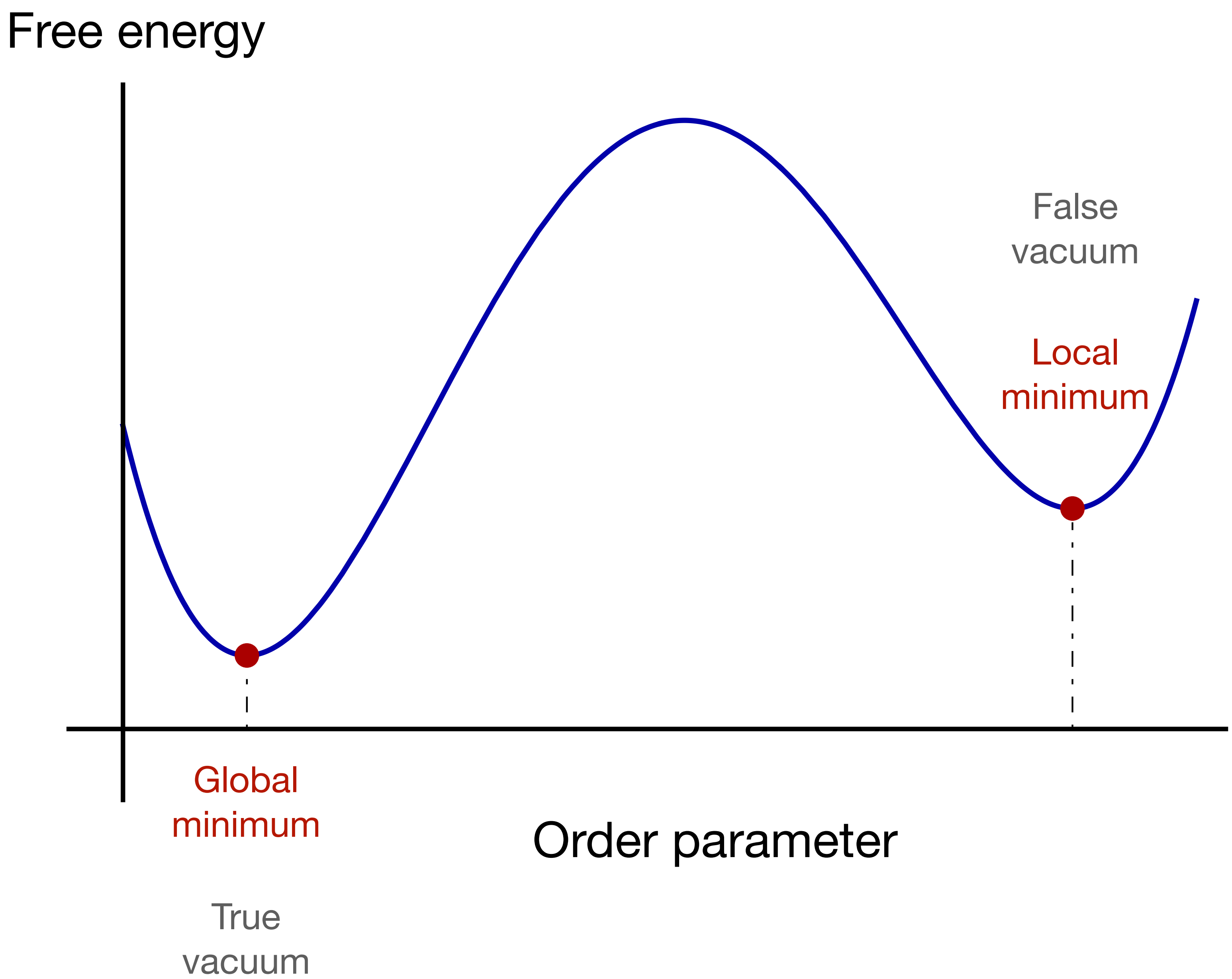
Miha Nemevšek
IJS & FMF



with Aleksandar Ivanov, Victor Guada, Alessio Maiezza, Marco Matteini, Yutaro Shoji and Lorenzo Ubaldi

International workshop on functional determinants,
Log pod Mangartom, 28th January 2024

Phase transitions

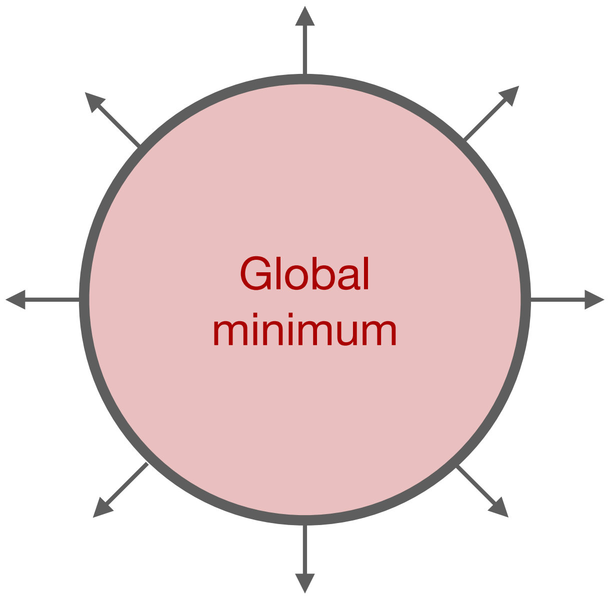


First order phase transition

Decay rate

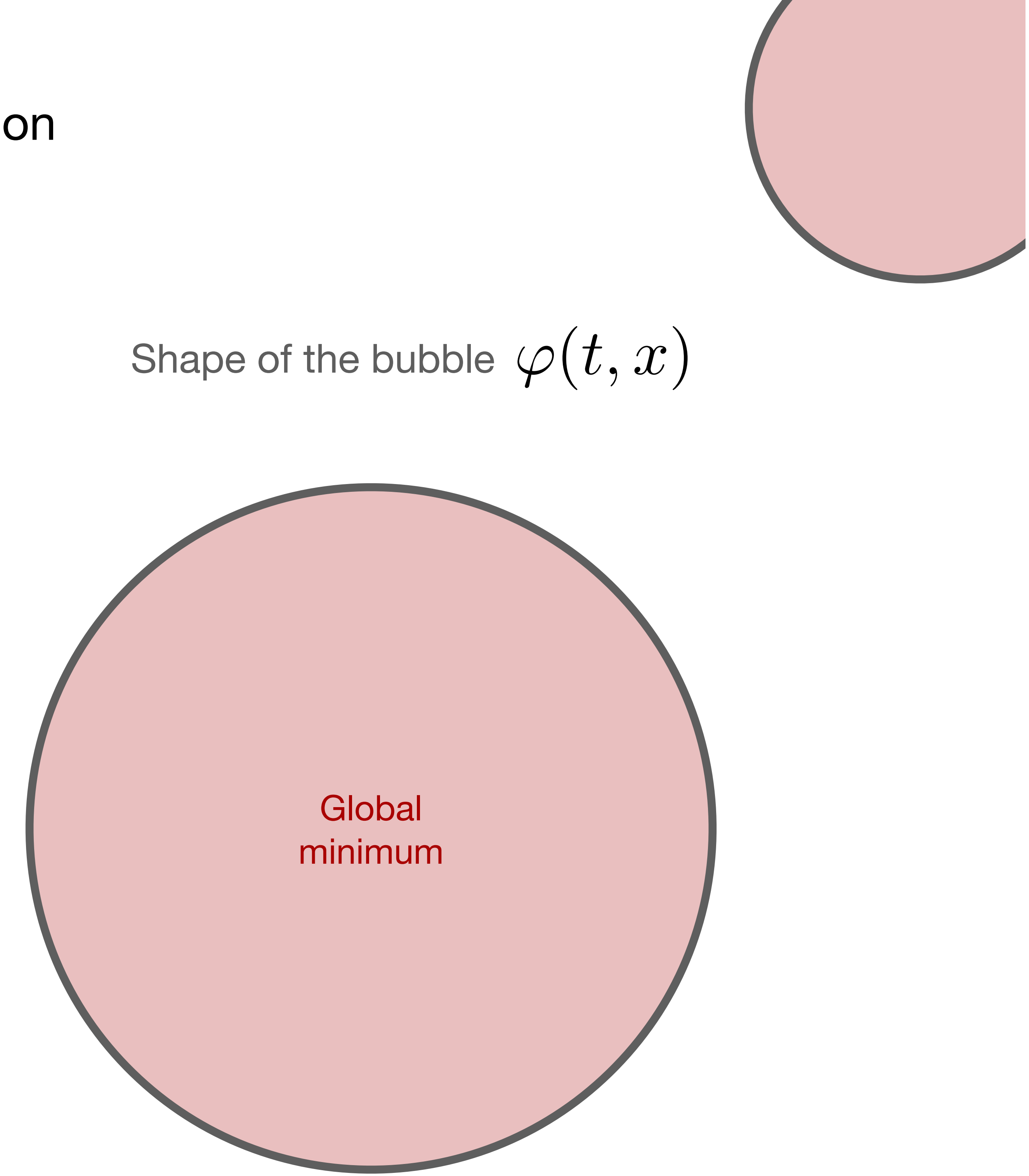
$$\Gamma = A\, e^{-\mathcal{S}}$$

Local
minimum



First order phase transition

Local
minimum

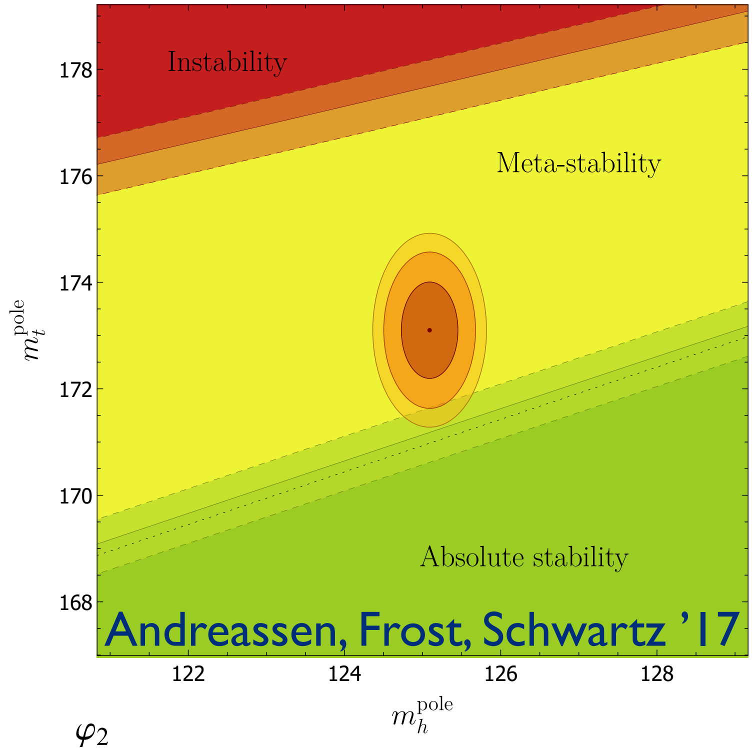


Physics

Quantum fluctuations

SM lifetime vs. $m_t,\ \alpha_s,\ m_h$

BSM stability - m_S, λ_i
selection of vacuua (landscape)

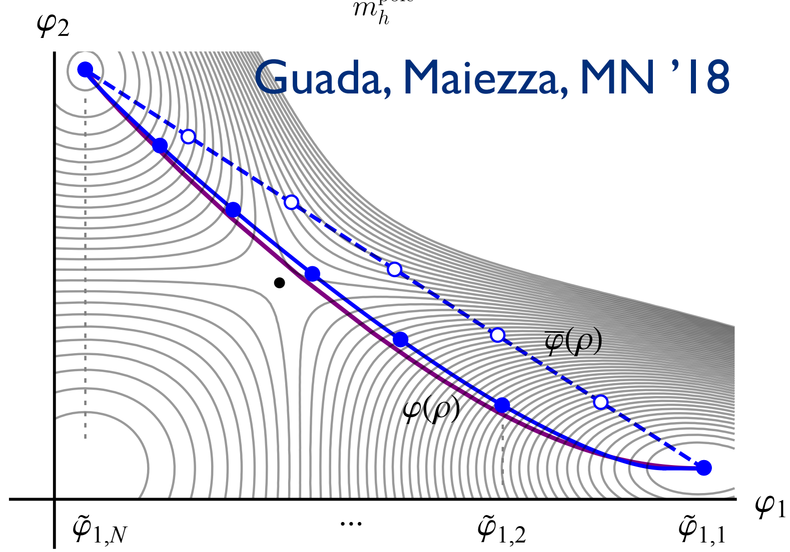


Thermal fluctuations

Gravitational waves, primordial *B*-fields

Baryogenesis / bubble wall fermion conversion

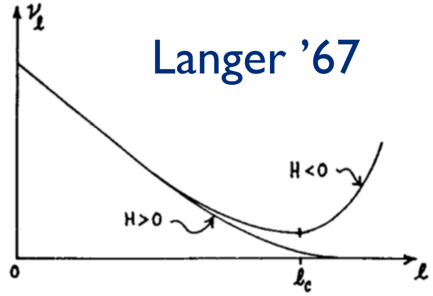
Inflation, dark energy, condensed matter, chemistry



a bit of History

Statistical physics

Theory of condensation point, droplet model, ferromagnets



Langer '67

QFT / Cosmology

Bubbles in Metastable Vacuum — Voloshin, Kobzarev, Okun '74

Fate of the false vacuum: semiclassical theory — Coleman '77

Fate of the false vacuum. II. First quantum corrections — Callan, Coleman '77

Gravitational effects on and of vacuum decay — Coleman, de Luccia '79

also T.D. Lee, G.C. Wick, I. Affleck, A. Linde, S. Hawking, …

Decay rate

Coleman '77

$$\mathcal{L} = \frac{1}{2}\partial\varphi^2 - V(\varphi) \qquad \Rightarrow \qquad \mathcal{L}_E = \frac{1}{2}\partial_E\varphi^2 + V(\varphi)$$

$$\frac{\Gamma}{\mathcal{V}} = \frac{\mathrm{Im}\int \mathcal{D}\varphi\, e^{-S[\varphi]}}{\int \mathcal{D}\varphi\, e^{-S[\varphi_{\rm FV}]}}$$

$$S[\varphi] \simeq S[\overline{\varphi}] + \underbrace{\left.\frac{\delta S}{\delta\varphi}\right|_{\overline{\varphi}}}_{=0}\delta\varphi + \frac{1}{2}\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\overline{\varphi}}\delta\varphi^2 + \dots$$

bounce extremize fluctuations

$O(4)$ Coleman, Glaser, Martin '78

$\rho^2 = t^2 + \sum x_i^2$ Euclidean time = radius of the bubble

Decay rate

Coleman '77

$$\mathcal{L} = \frac{1}{2}\partial\varphi^2 - V(\varphi) \qquad \Rightarrow \qquad \mathcal{L}_E = \frac{1}{2}\partial_E\varphi^2 + V(\varphi)$$

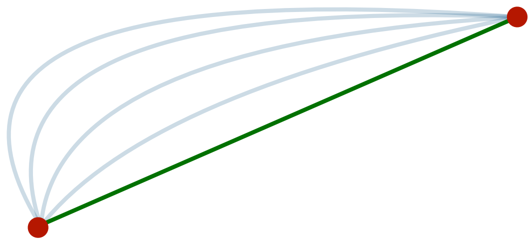
$$\frac{\Gamma}{\mathcal{V}} = \frac{\mathrm{Im}\int \mathcal{D}\varphi\, e^{-S[\varphi]}}{\int \mathcal{D}\varphi\, e^{-S[\varphi_{\mathrm{FV}}]}}$$

$$S[\varphi] \simeq S[\overline{\varphi}] + \underbrace{\left.\frac{\delta S}{\delta\varphi}\right|_{\overline{\varphi}}}_{=0}\delta\varphi + \frac{1}{2}\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\overline{\varphi}}\delta\varphi^2 + \ldots$$

"...there always exists an *O(4)*-invariant bounce and it always has strictly lower action than any non-*O(4)* invariant bounce. The rigor of our proof is matched only by its tedium; I wouldn't lecture on it to my worst enemy." Coleman, Erice lectures '77

multi-fields

Blum, Honda, Sato, Takimoto, Tobioka '16



$$\delta S = 0 \to \overline{\varphi}(\rho)$$

Bounce

* Thin wall — Ivanov, Matteini, MN, Ubaldi ’22
* Polygonal bounces — Guada, Maiezza, MN ’18
* `FindBounce` — Guada, MN, Pintar ’20

The bounce *D* dimensional symmetric Euclidean action

$$S = \Omega \int_0^\infty \mathrm{d}\rho\, \rho^{D-1} \left(\frac{1}{2}\dot{\phi}^2 + V - V_{\rm FV} \right) , \qquad \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

bounce equation

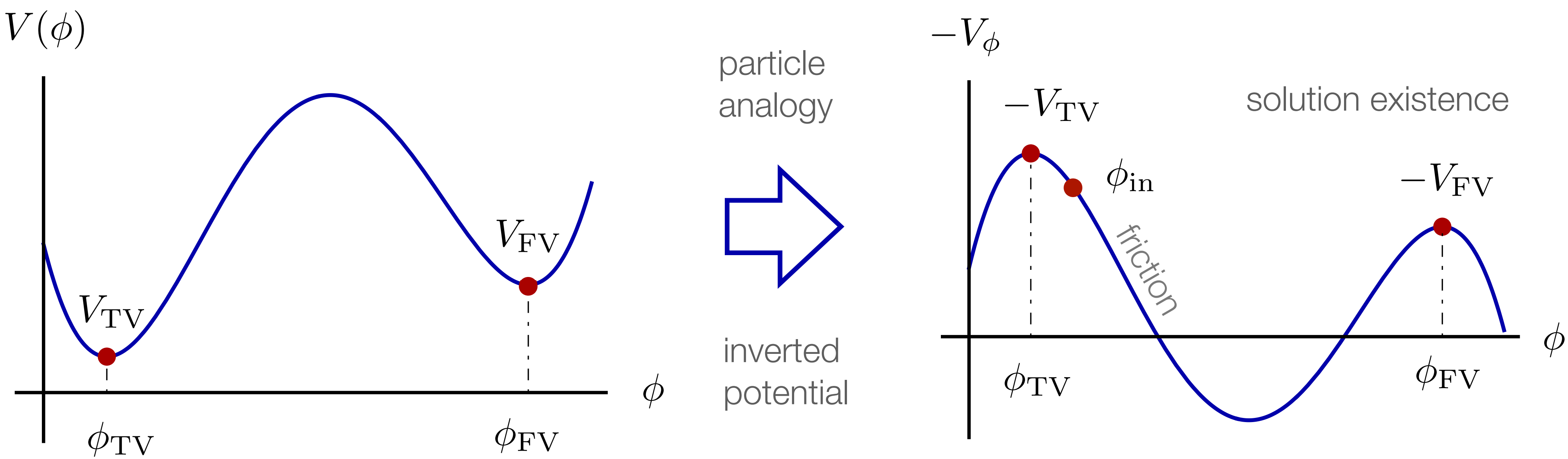
$$\ddot{\phi} + \frac{D-1}{\rho}\dot{\phi} = \frac{\mathrm{d}V}{\mathrm{d}\phi}$$

friction

boundary conditions

$$\dot{\phi}(0) = \dot{\phi}(\infty) = 0 \,,$$

$$\phi(0) = \phi_{\rm in} \,, \qquad \phi(\infty) = \phi_{\rm FV}$$



Thin wall and beyond

Thin Wall

Ivanov, Matteini, MN, Ubaldi ’22

$$V = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 (\phi - v)$$

Overall coupling

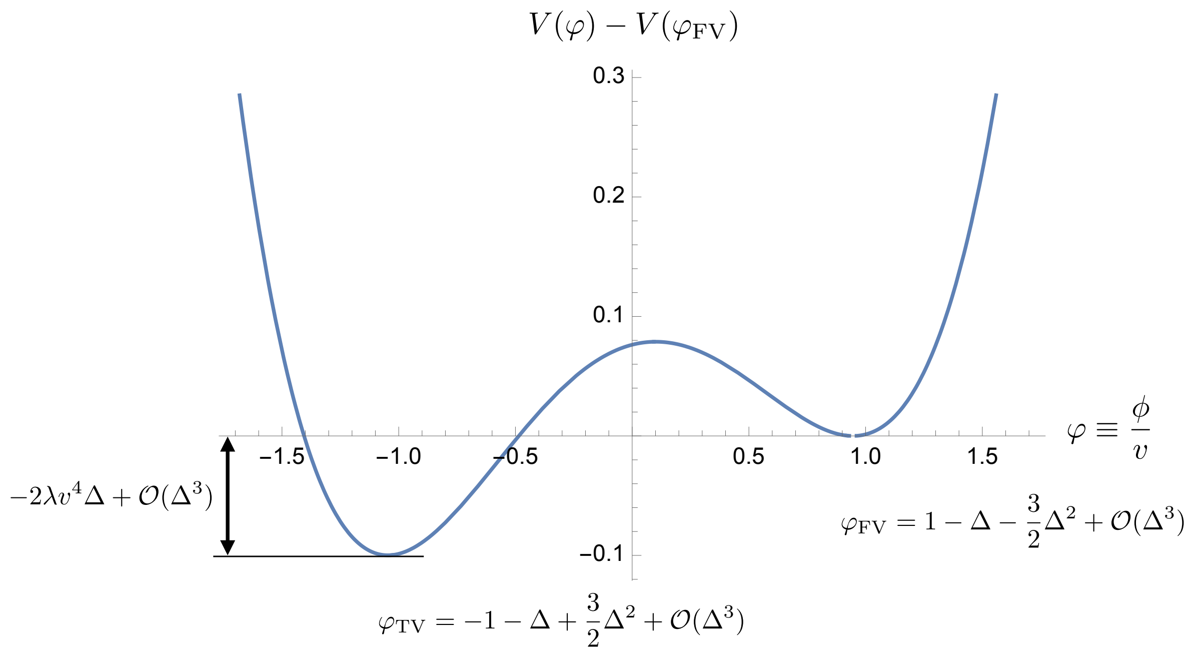
$[\lambda] = 4 - D$

Scale of the model

$[v] = D/2 - 1$

TW parameter

$[\Delta] = 0$



Perturbative

$$0 < \lambda \ll 1, \quad 0 < \Delta \ll 1, \quad \Delta_{\rm max} = 3^{-3/2}$$

TW = near degenerate

$$V = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 (\phi - v)$$

Ivanov, Matteini, MN, Ubaldi '22

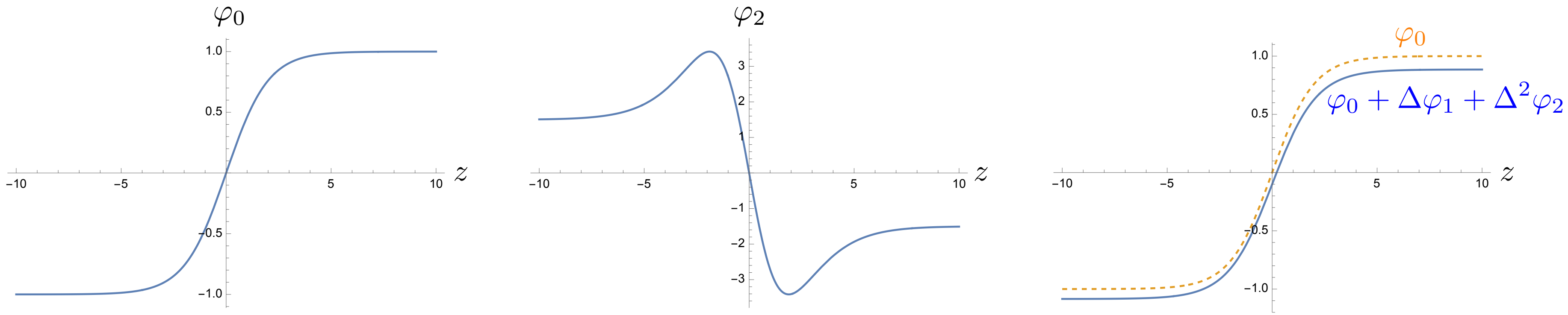
Expand the field around the bounce radius

$$\varphi(z) = \sum \varphi_n(z) \Delta^n , \qquad z = \sqrt{\lambda} v \, \rho - r , \qquad r = \frac{1}{\Delta} \sum r_n \Delta^n$$

We solve the bounce equation

$$\varphi_0 = \mathrm{th}\frac{z}{2} , \qquad \varphi_1 = -1 ,$$

$$r_0 = \frac{D-1}{3} , \qquad r_1 = 0$$



See the paper for φ_2, φ_3, physical bubble profile

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}$$

Thin Wall Action

Ivanov, Matteini, MN, Ubaldi '22

λ and v factor out

$$S = \frac{\Omega\, v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \times \tilde{S}[\Delta^2]$$

Leading order

$$S_0 = \frac{\Omega\, v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \left(\frac{D-1}{3}\right)^{D-1} \frac{2}{3D}$$

Matteini, MN, Shoji, Ubaldi '23

Now also done + more

Higher orders

$$S_2 = S_0 \left(1 + \Delta^2 \left(\frac{1 + D\left(25 - 8D - 3\pi^2\right)}{2(D-1)}\right) + \mathcal{O}(\Delta^4)\right)$$

New & relevant

Perturbativity

$D = 4$: FV decay at $T = 0$ — Coleman '77

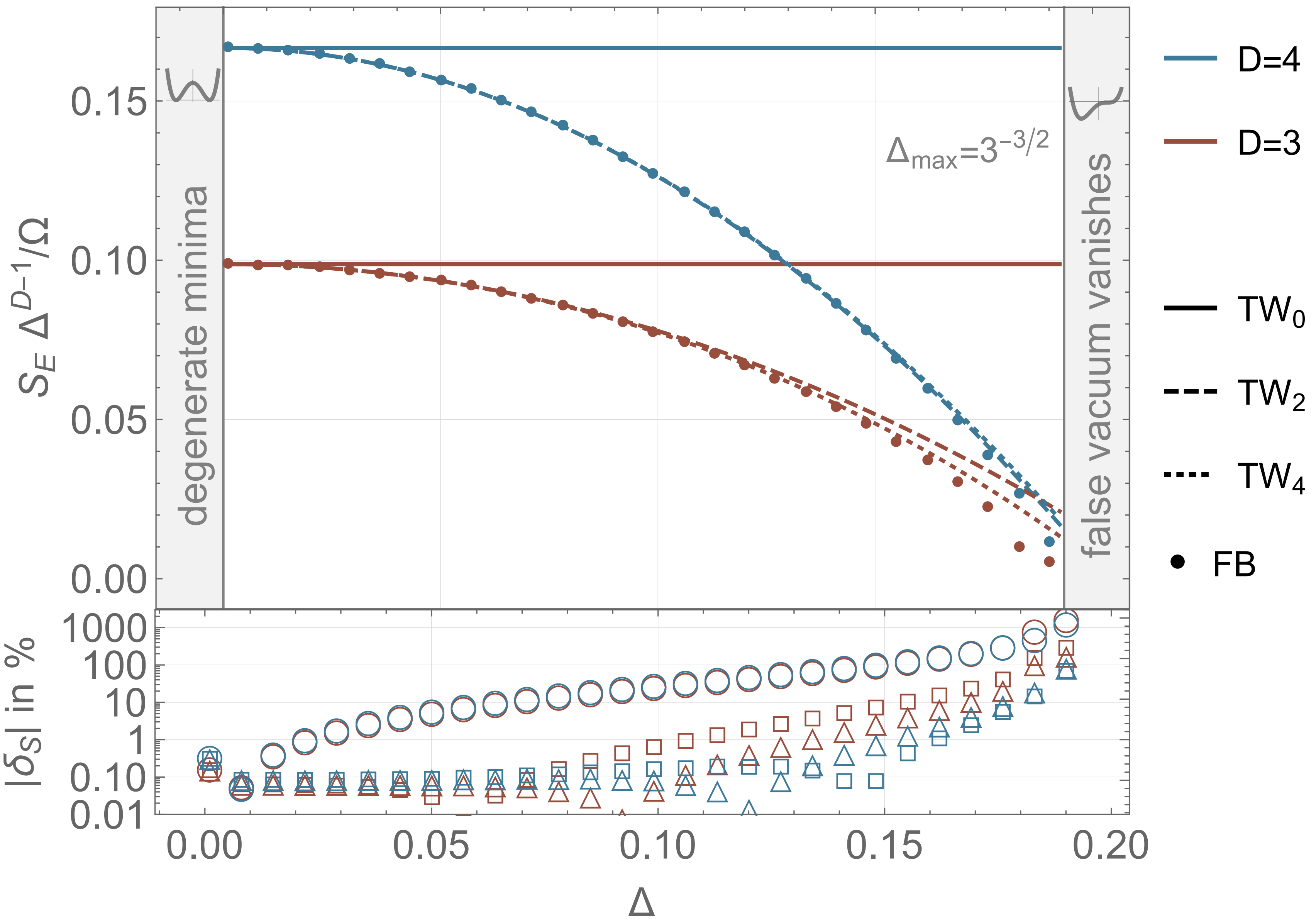
$D = 3$: FV nucleation at finite T — Affleck '81, Linde '83

$$S_2 = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1\right)\Delta^2\right), & D = 3, \\ \frac{\pi^2}{3\lambda} \left(1 - \left(2\pi^2 + \frac{9}{2}\right)\Delta^2\right), & D = 4 \end{cases}$$

Thin Wall Action

Ivanov, Matteini, MN, Ubaldi '22

How good/useful is TW? Pretty good all the way to $\Delta_{\max}$!



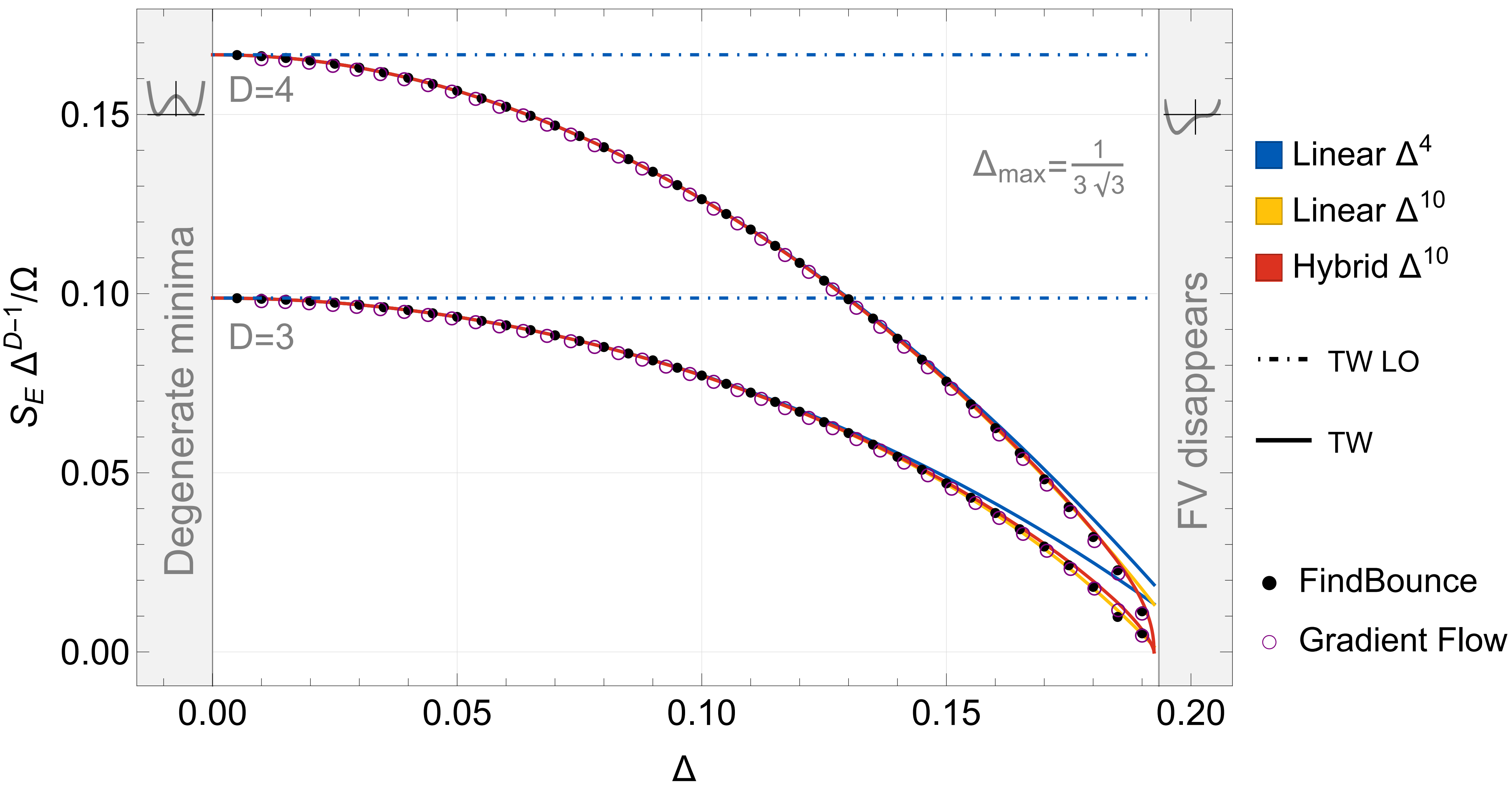
Higher orders: analytic Δ^4, semi-analytic up to Δ^{10}

Matteini, MN, Shoji, Ubaldi '23

Thin -> thick Wall Action

Matteini, MN, Shoji, Ubaldi '23

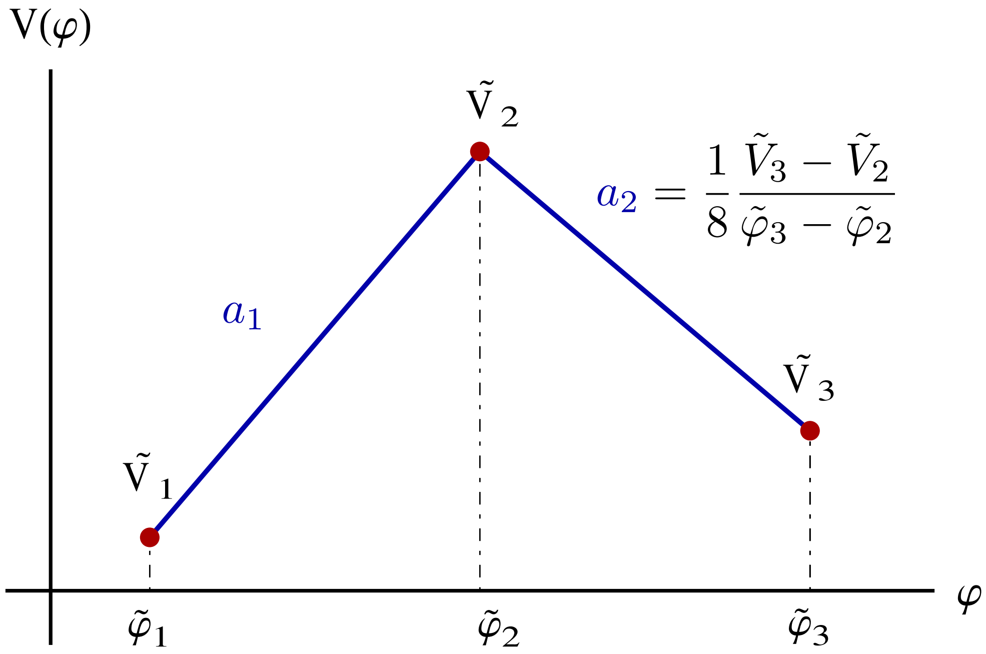
Linear and cubic potentials different outside TW



Possible to create a hybrid approach, good near inflection

Polygonal bounces

Triangle



Linear potentials

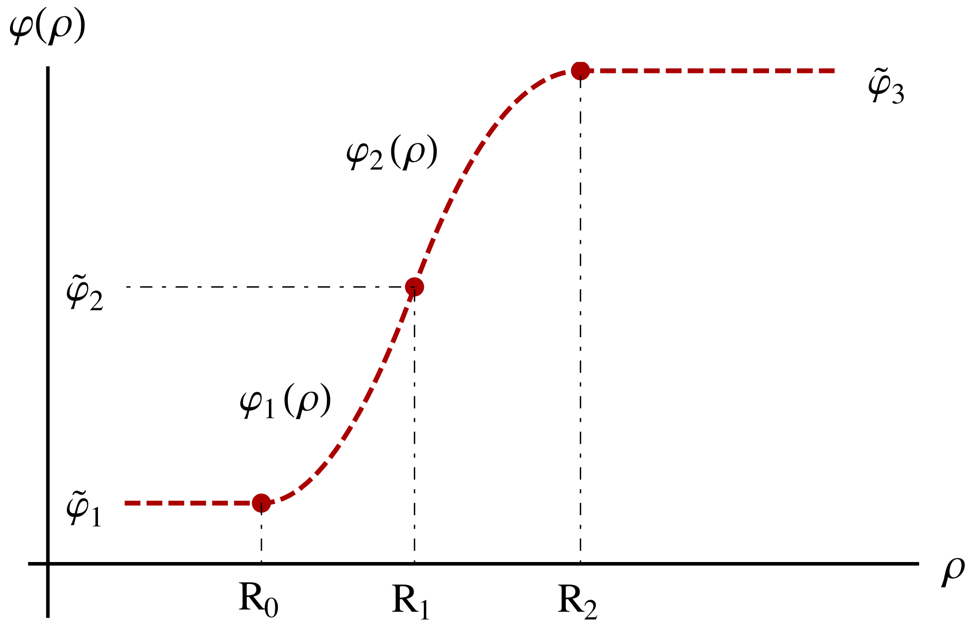
Duncan, Jensen '92

- triangle and box

Exact solution

$$\ddot{\varphi} + \frac{3}{\rho}\dot{\varphi} = dV = 8\,a$$

$$\varphi = v + a\,\rho^2 + \frac{b}{\rho^2}$$



Initial conditions @ R_0

shoot in φ_0 or R_0

- a) $\varphi_1(0) = \varphi_0, \quad \dot{\varphi}_1(0) = 0$

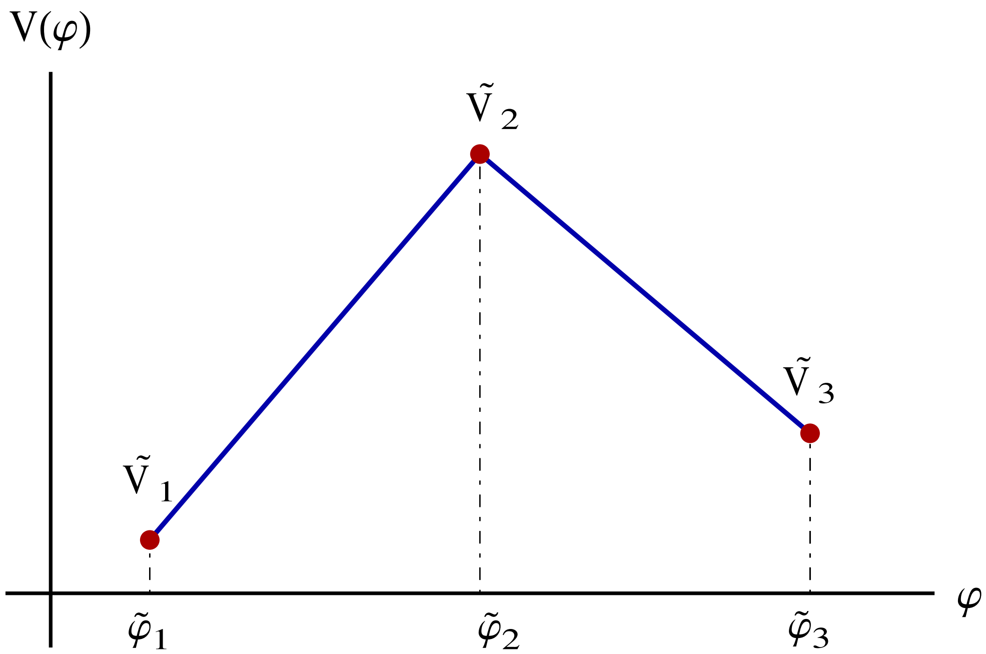
  $v_1 = \varphi_0, \quad b_1 = 0$

- b) $\varphi_1(R_0) = \tilde{\varphi}_1, \quad \dot{\varphi}_1(R_0) = 0$

  $v_1 = \tilde{\varphi}_1 - 2\,a_1\,R_0^2, \quad b_1 = a_1\,R_0^4$

Triangle summary

Duncan, Jensen '92



Complete exact analytic solution in *D=4* for two segments

Solved in terms of Euclidean radius

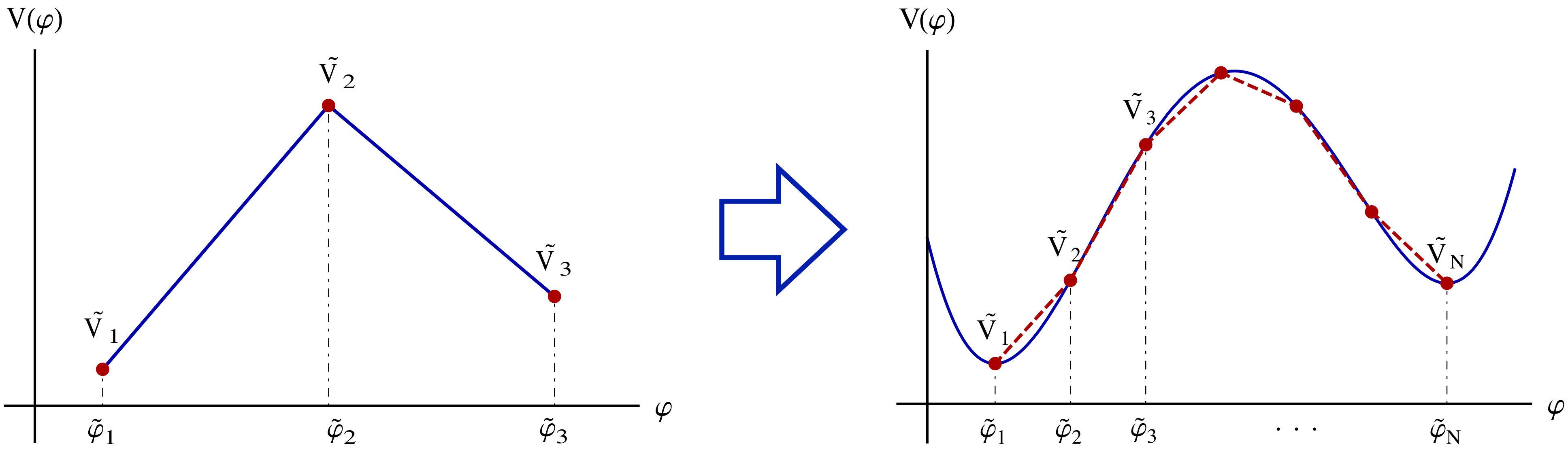
Stable in thin wall, goes over to TW

Limited validity outside the TW

Polygonal bounces

Extend to more segments and *D* dimensions

Guada, Maiezza, MN ’18

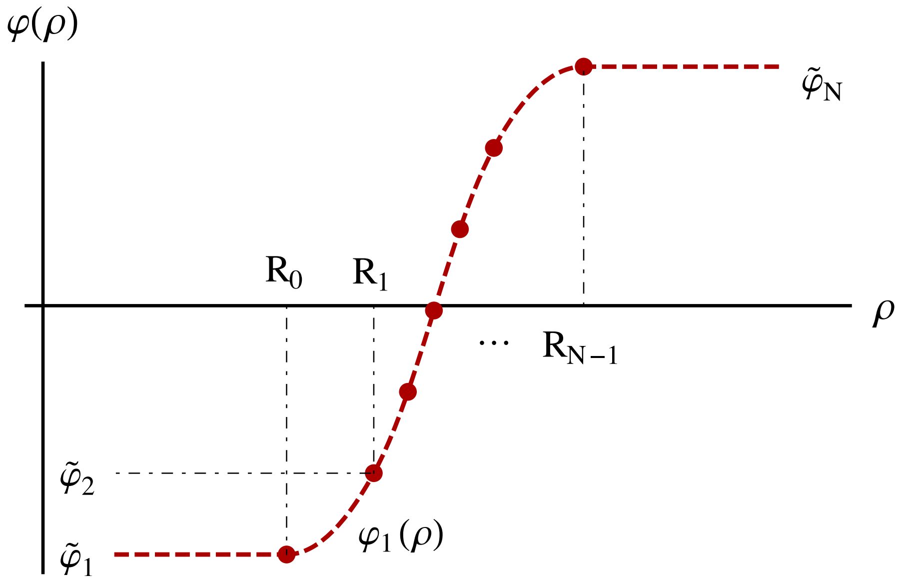


Approximates any V when $N \to \infty$, with controlled precision

Geometric insight of segmentation, cover non-trivial features/unstable Vs

Semi-analytic solution for algebraic manipulation/deformation

Polygonal construction



$$\ddot\varphi_i + \frac{D-1}{\rho}\dot\varphi_i = \mathrm{d}V_i = 8\,a_i$$

$$\varphi_i = v_i + \frac{4}{D}a_i\,\rho^2 + \frac{2}{D-2}\frac{b_i}{\rho^{D-2}}$$

Initial/final conditions remain the same

Matching conditions @ R_i 3 parameters and 3 unknowns/segment **Guada, Maiezza, MN '18**

$$\varphi_i(R_1) = \varphi_{i+1}(R_i) = \tilde\varphi_{i+1}, \quad \dot\varphi_i(R_i) = \dot\varphi_{i+1}(R_i)$$

The bounce is defined recursively

- a) $R_0 = 0$

$$v_n = \varphi_0 - \frac{4}{D-2}\left(a_1 R_0^2 + \sum_{i=1}^{n-1}(a_{i+1}-a_i)\,R_i^2\right)$$

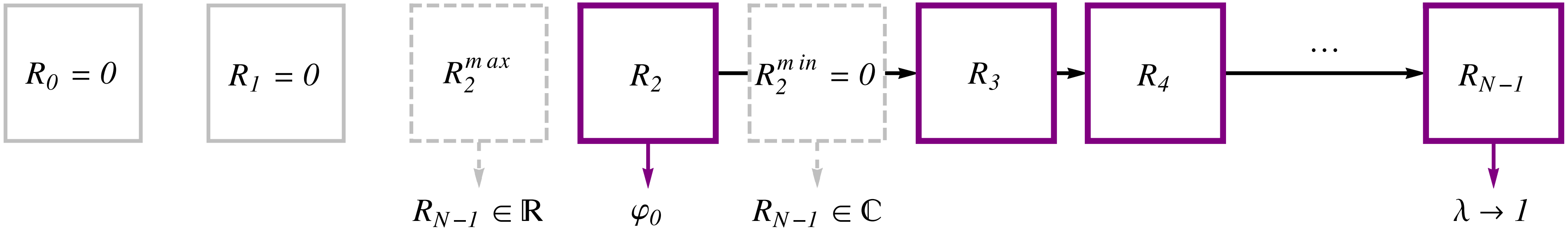
- b) $\varphi_0 = \tilde\varphi_1$

$$b_n = \frac{4}{D}\left(a_1 R_0^D + \sum_{i=1}^{n-1}(a_{i+1}-a_i)\,R_i^D\right)$$

Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

fewnomial for R_n $$\boxed{R_n^D} - \frac{D}{4}\frac{\delta_n}{a_n}\boxed{R_n^{D-2}} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0$$ $\delta_n = \tilde{\varphi}_{n+1} - v_n$

require real positive roots



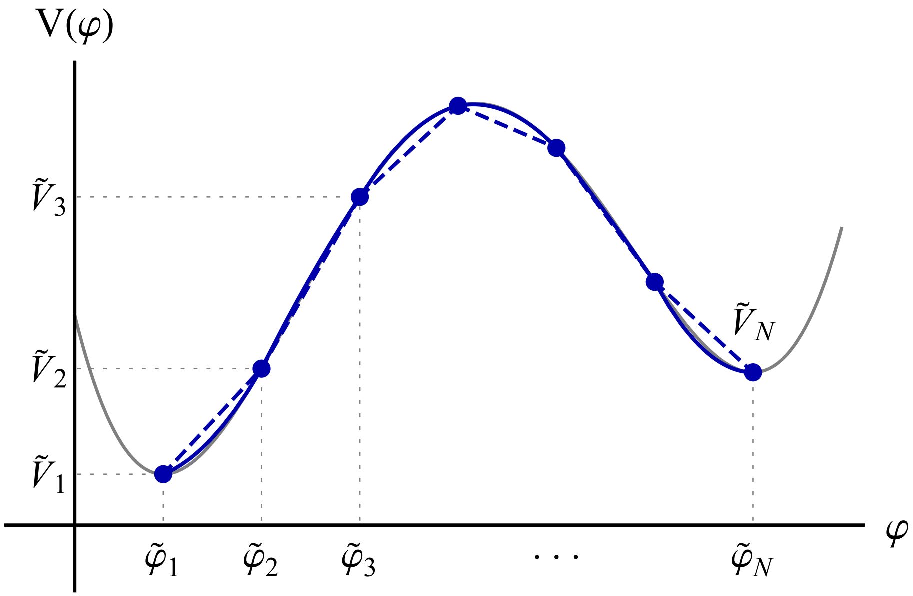
radii solutions $$D = 3: \quad 2R_n = \frac{1}{\sqrt{a_n}}\left(\frac{\delta_n}{\xi} + \xi\right), \qquad \xi^3 = \sqrt{36 a_n b_n^2 - \delta_n^3} - 6\sqrt{a_n} b_n,$$

simple cubic

$$D = 4: \quad 2R_n^2 = \frac{1}{a_n}\left(\delta_n + \sqrt{\delta_n^2 - 4 a_n b_n}\right)$$ quadratic

$D = 2, 6, 8$ in the paper, other *D*s possible numerically Amariti '20

Higher orders



Expand to higher orders

- improves convergence
- important @ extrema

$$V_i \simeq \tilde{V}_i - \tilde{V}_N + \partial \tilde{V}_i \left(\varphi_i - \tilde{\varphi}_i\right)$$

$$+ \frac{\partial^2 \tilde{V}_i}{2} \left(\varphi_i - \tilde{\varphi}_i\right)^2$$

Perturbative expansion $\quad \varphi = \varphi_{PB} + \xi$

$$\ddot{\varphi} + \frac{D-1}{\rho}\dot{\varphi} = 8\left(a + \alpha\right) + \delta dV\left(\varphi_{PB}(\rho)\right)$$

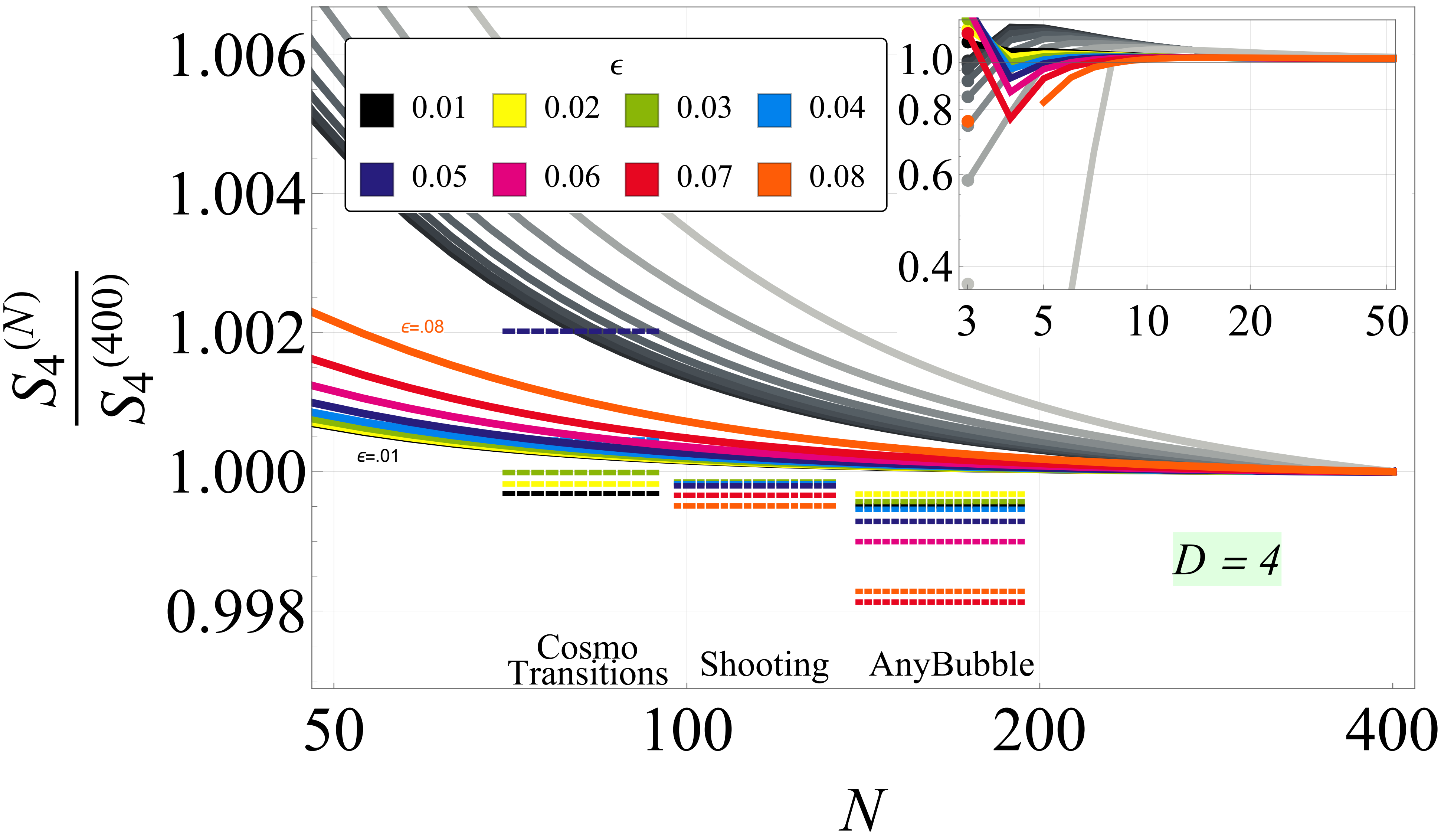
$$\ddot{\xi} + \frac{D-1}{\rho}\dot{\xi} = 8\alpha + \delta dV(\rho)$$

$$\mathcal{I}_s^{D=4} = \partial^2 \tilde{V}_s \left(\frac{v_s - \tilde{\varphi}_s}{8}\rho^2 + \frac{b_s}{2}\ln\rho + \frac{a_s}{24}\rho^4 \right)$$

$$\xi = \nu + \frac{2}{D-2}\frac{\beta}{\rho^{D-2}} + \frac{4}{D}\alpha\rho^2 + \mathcal{I}(\rho)$$

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right)$$

Guada, Maiezza, MN ’18

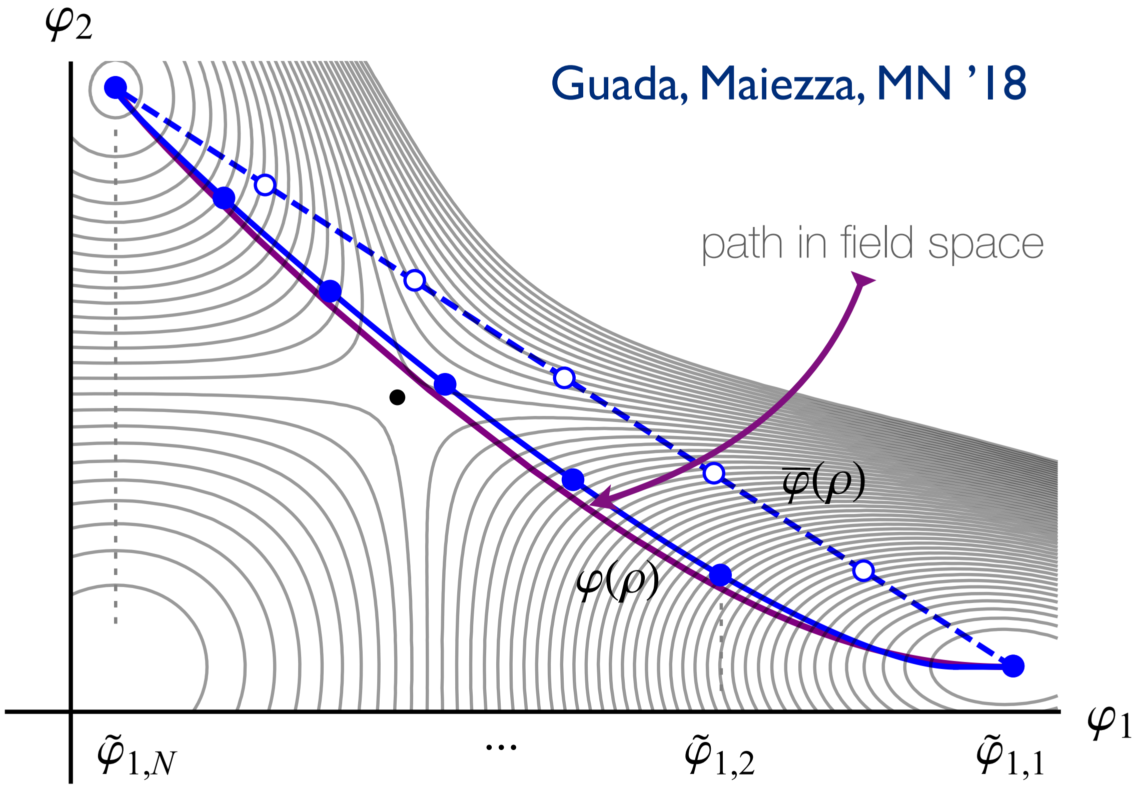


Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{\mathrm{d}V}{\mathrm{d}\varphi_i} \qquad \varphi_i(0) = \varphi_{0i}$$

- CosmoTransitions Wainwright '11

  bounce and path deformation separate, oscillations, Runge-Kutta PDE solver



- AnyBubble Masoumi, Olum, Shlaer '16

  multiple shooting, damping approximations

- Other recent approaches

  tunnelling potential

  Espinosa, Konstandin '18

  machine learning

  Piscopo, Spannowsky, Waite '19

  gradient flow

  Sato '19

Polygonal approach with many fields

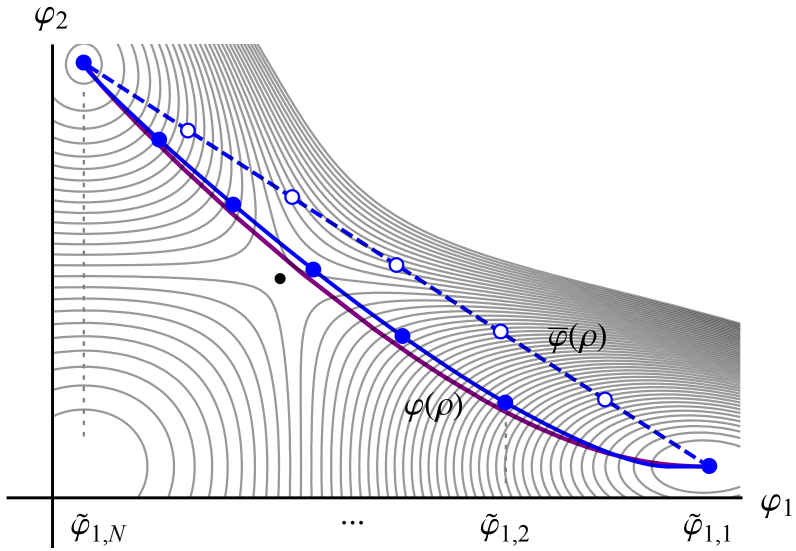
Guada, Maiezza, MN '18

- Initial ansatz straight line, via saddle, custom segmentation

- Initial solution longitudinal single field PB

$\tilde{\varphi}_{is}$

$\bar{\varphi}_{is}(\rho)$



Crucial idea #1

- perturbation up to linear term in V, keeps the PB

$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{\mathrm{d}V}{\mathrm{d}\varphi_i}(\bar{\varphi}+\zeta)$$

$$\zeta_{is} = v_{is} + \frac{2}{D-2}\frac{b_{is}}{\rho^{D-2}} + \frac{4}{D}a_{is}\rho^2$$

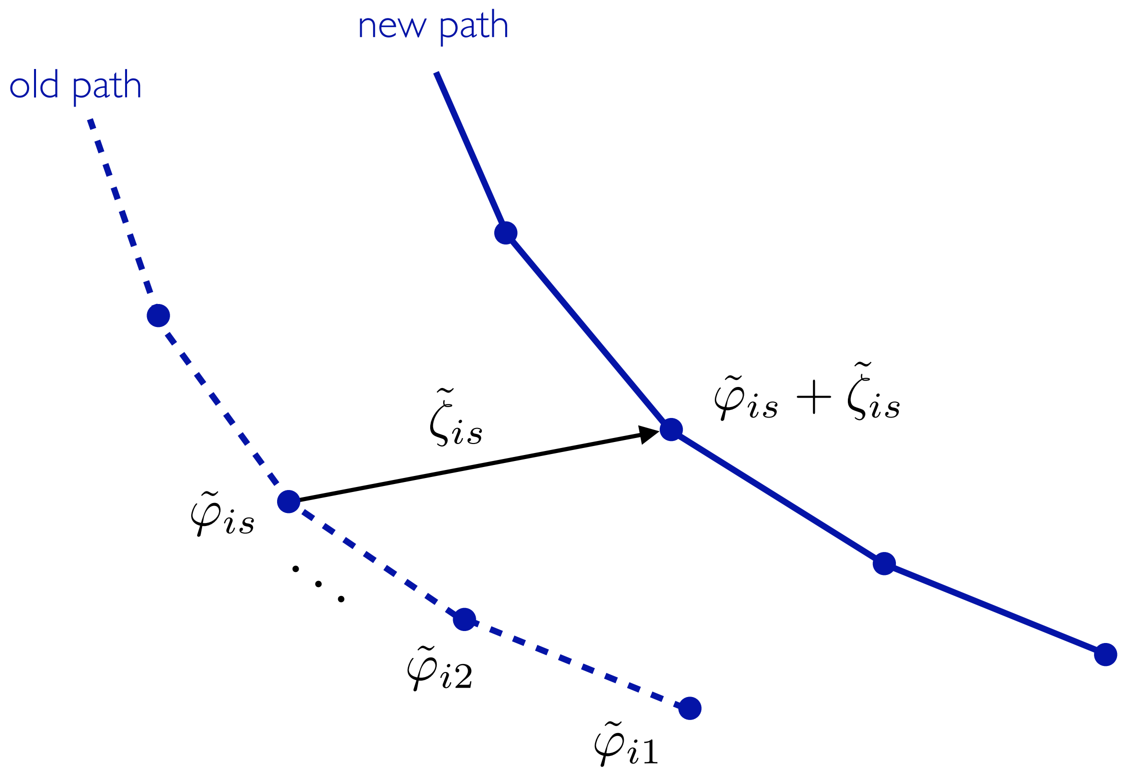
$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{\mathrm{d}V}{\mathrm{d}\varphi_i}(\bar{\varphi} + \zeta)$$ Guada, Maiezza, MN '18

Crucial idea #2

$$8a_{is} \simeq \frac{dV}{d\varphi_i}\left(\tilde{\varphi}_{is} + \tilde{\zeta}_{is}\right) - 8\bar{a}_{is}$$

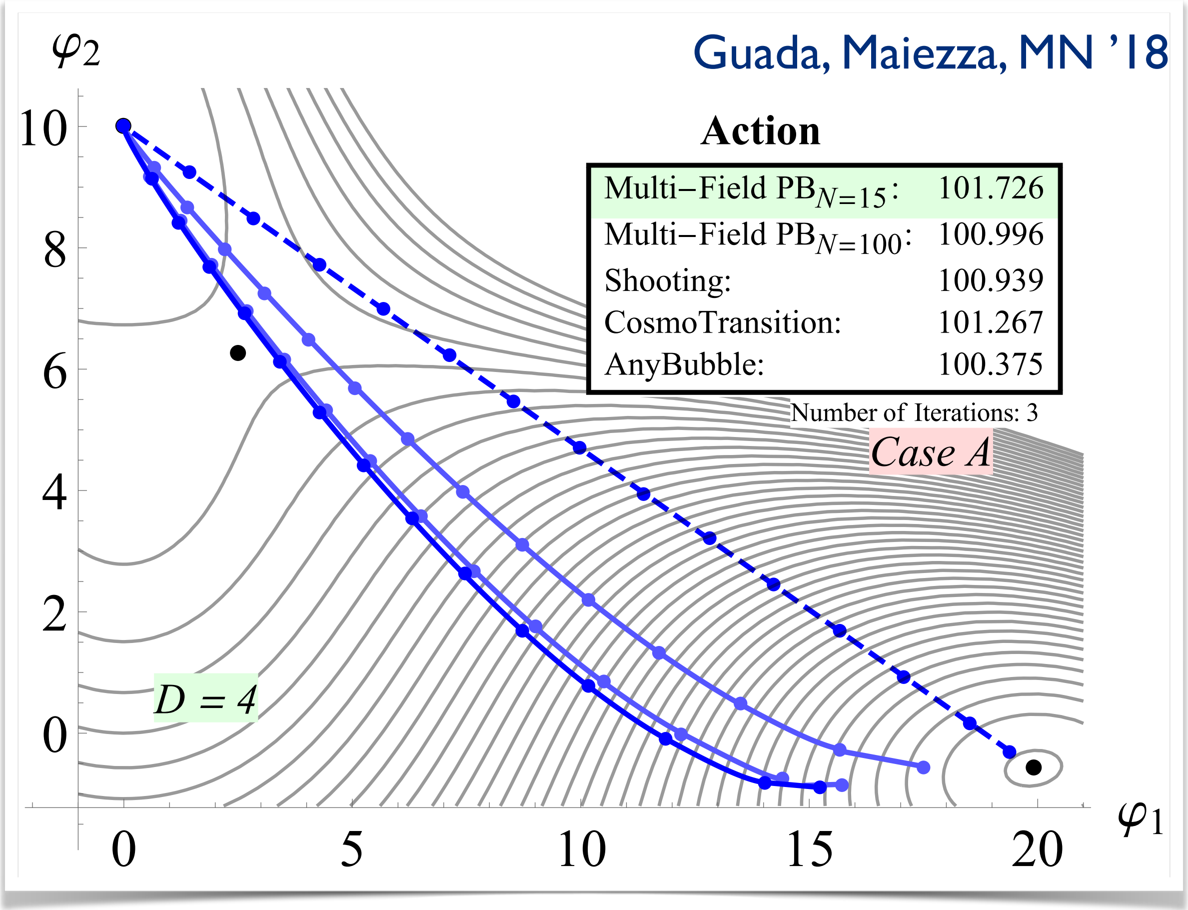
$$\frac{dV}{d\varphi_i} \simeq \frac{1}{2}\left(d_i\tilde{V}_s + d_i\tilde{V}_{s+1} + d^2_{ij}\tilde{V}_s\tilde{\zeta}_{js} + d^2_{ij}\tilde{V}_{s+1}\tilde{\zeta}_{js+1}\right)$$

- simultaneous solution for the bounce and path deformation
- linear system for r_{i0} (as in the single field expansion) and $\tilde{\zeta}_{is}$
- iterate until $\tilde{\zeta}_{is} < \varepsilon_{\Delta\varphi}$



$$V(\varphi_i) = \sum_{i=1}^{2} \left(-\mu_i^2 \varphi_i^2 + \lambda_i^2 \varphi_i^4\right) + \lambda_{12} \varphi_1^2 \varphi_2^2 + \tilde{\mu}^3 \varphi_2$$

- no oscillations
- converges in a few iterations
- works for thin wall
- works for *D*=3 and *4*
- tested for up to 20 fields



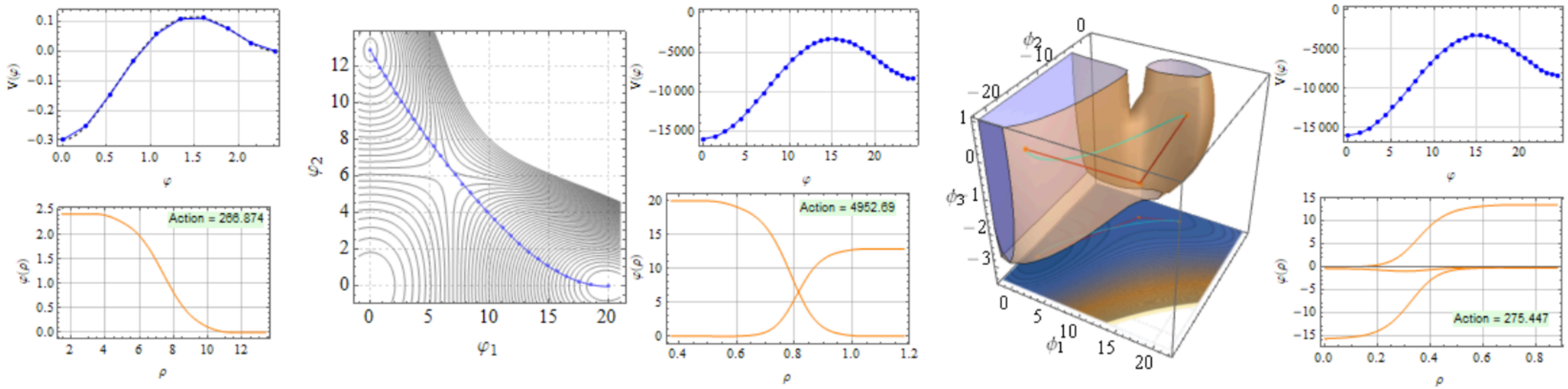
FindBounce

Guada, MN, Pintar '20

FindBounce https://github.com/vguada/FindBounce/releases

FindBounce is a Mathematica package that computes the bounce configuration needed to compute the false vacuum decay rate with multiple scalar fields.

We kindly ask the users of the package to cite the two papers that describe the working of the *FindBounce* package: the paper with the original proposal by Guada, Maiezza and Nemevšek (2019) and the software release manual by Guada, Nemevšek and Pintar (2020).



Installation

To use the *FindBounce* package you need Mathematica version 10.0 or later. The package is released in the `.paclet` file format that contains the code, documentation and other necessary resources. Download the latest `.paclet` file from the repository "releases" page to your computer and install it by evaluating the following command in the Mathematica:

```
(* Path to .paclet file downloaded from repository "releases" page. *)
PacletInstall["full/path/to/FindBounce-X.Y.Z.paclet"]
```

Load the package as usual

Guada, MN, Pintar ’20

```
In[1]:= Needs["FindBounce`"]
```

Define a metastable potential

```
In[2]:= V[x_]:= 0.5 x^2 + 0.5 x^3 + 0.12 x^4;

In[3]:= extrema = x/.Sort@Solve[D[V[x],x]==0];
```

Compute the bounce - obtain bf = the bounce function

```
In[4]:= bf = FindBounce[V[x],x,{extrema[[1]],extrema[[3]]}]

Out[4]= BounceFunction[ + 
```

```
 Action: 73500.
 Dimension: 4 ]

In[5]:= bf["Action"]

Out[5]= 73496.

In[6]:= bf["Dimension"]

Out[6]= 4
```

Retrieve the bounce properties

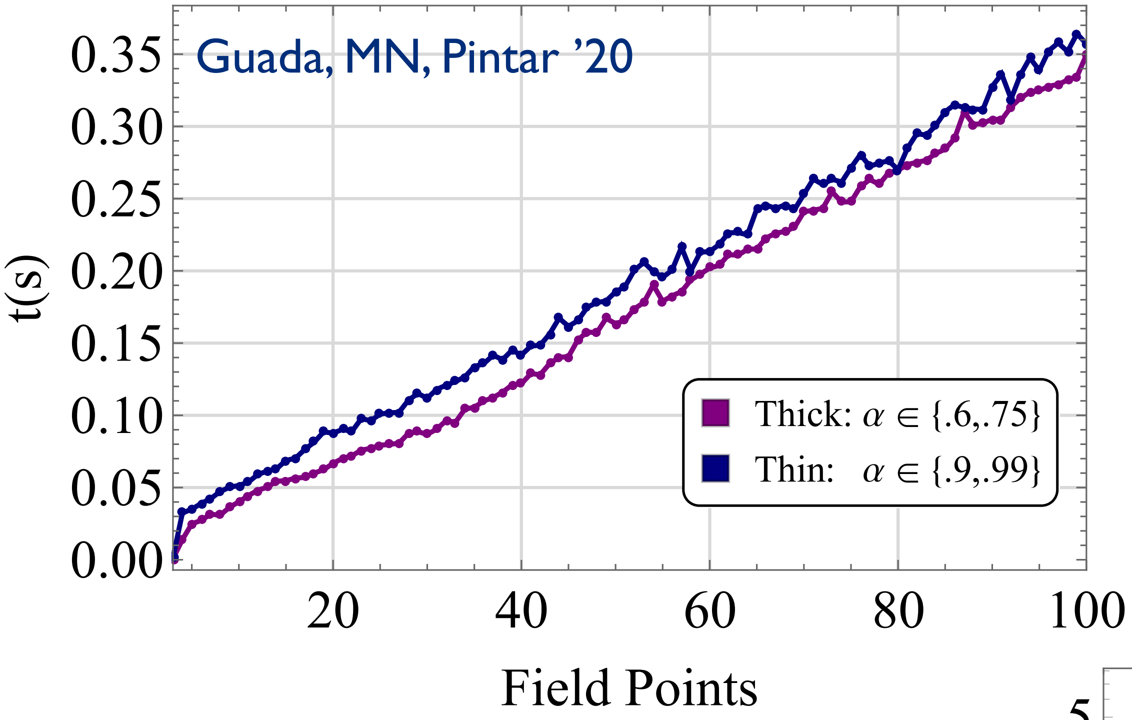
Run on single points

```
FindBounce[{{x1,V1},{x2,V2},...}]
```

And multifields

```
FindBounce[V[x,y,...],{x,y,...},{m1,m2}]
```

Time demand



Scales linearly by construction

Works in thin and thick regimes

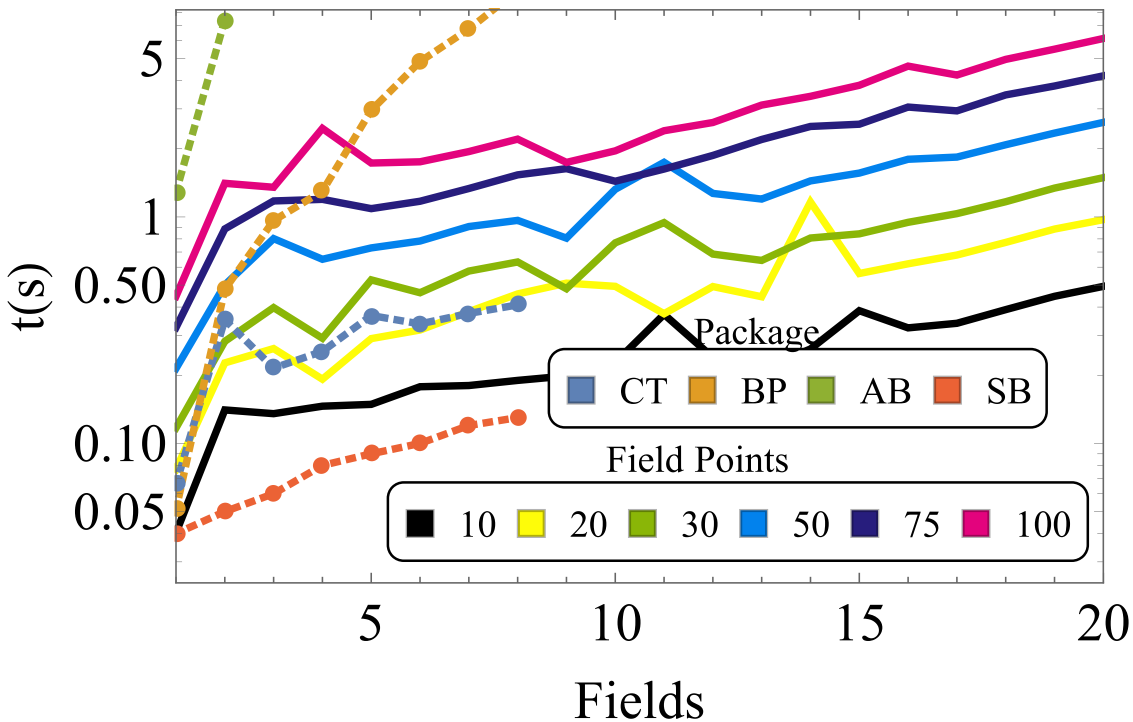
Tested up to 20 fields

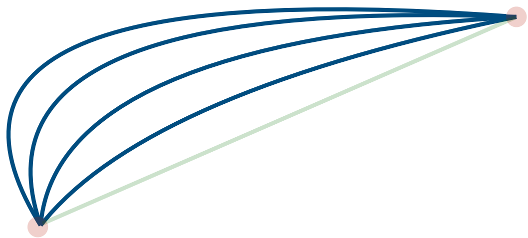
CT - CosmoTransitions — Wainwright '11

BP - BubbleProfiler — Masoumi et al. '16

AB - AnyBubble — Athron et al. '19

SB - SimpleBounce — Sato '20





$$\int \mathcal{D}\varphi \to \prod \lambda_i$$

Prefactors

- Thin wall — Ivanov, Matteini, MN, Ubaldi ’22
- Polygonal bounce — Guada, Maiezza, MN ’18
- Quartic-quartic — Guada, MN ’20

TW rate at one loop - introduction

Coleman '88

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{S_R}{2\pi\hbar}\right)^{\frac{D}{2}} \left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S_{\rm ct}} \left(1 + \mathcal{O}(\hbar)\right)$$

$\mathcal{O} = -\partial_\mu \partial^\mu + V^{(2)}$ fluctuations around the bounce

Bubble deformation, zeroes for symmetries, single negative

a) Renormalized bounce action and counter-terms

b) Functional determinant

Pedagogical notes on functional determinants

Dunne '07

c) Zero removal

$\det' \mathcal{O}$

a) Renormalized bounce action

Guada, Ivanov, MN, Ubaldi '21

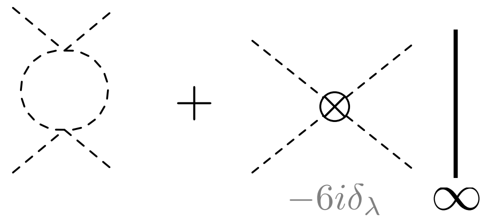
At one loop, the action needs counter-terms

Peskin & Schröder

$$V_{\rm ct} = \frac{\delta_{m^2}}{2}\phi^2 + \frac{\delta_\lambda}{4}\phi^4, \qquad \langle\phi\rangle = \frac{\mu}{\sqrt{\lambda}}\,, \qquad V^{(n)} \equiv \frac{{\rm d}^n V}{{\rm d}\phi^n}(\langle\phi\rangle)$$

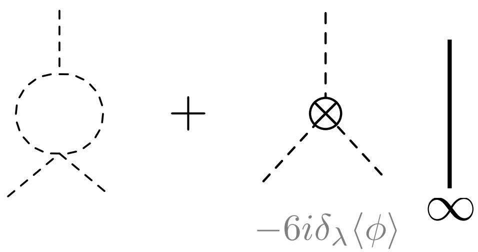
$$\delta_\lambda = \frac{1}{32\pi^2\varepsilon}V^{(4)2}$$

set by the 4-p function



$= 0 \qquad \varepsilon = 4 - D$

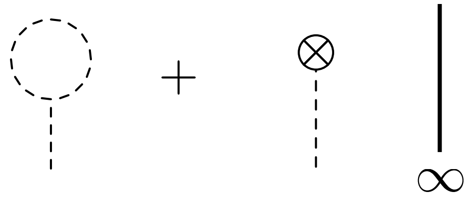
cancels the 3-p



$= 0$ if $V^{(3)} = V^{(4)}\langle\phi\rangle$

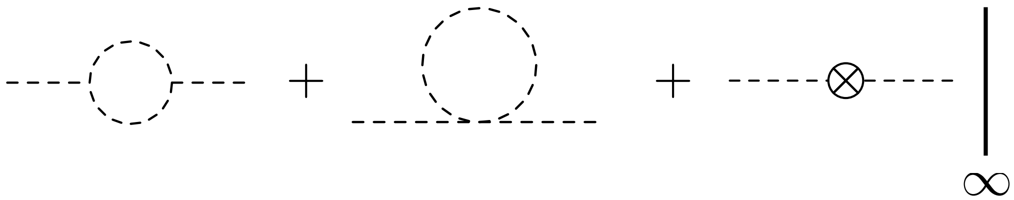
automatic if 3-p interactions comes from a single quartic

$$V^{(4)} = 4!\,\lambda, \qquad V^{(3)} = 4\times 3!\,\lambda\langle\phi\rangle = V^{(4)}\langle\phi\rangle$$

Remove the tadpoles…  $= 0$

$$\delta_{m^2} = \frac{1}{(4\pi)^2\,\varepsilon} V^{(4)} \left(V^{(2)} - \frac{1}{2} V^{(4)} \langle\phi\rangle^2 \right)$$

…and the 2-p mass divergence cancels out

 $= 0$

$$\langle\phi\rangle \simeq v\,(1 - \Delta + \ldots)$$

the vev shifts

$$\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2\,2\varepsilon}\,, \qquad \delta_{m^2} = -\frac{3\lambda^2 v^2}{(4\pi)^2\,2\varepsilon}$$

independent of Δ

Δ counter-term is zero, and does not run (2 loops?)

a) Renormalized bounce action and counter-terms

Counter-term for the Euclidean action

$$S_{\text{ct}} = \int_D (V_{\text{ct}} - V_{\text{ctFV}}) = \frac{3\lambda^2}{8(4\pi)^2 \varepsilon} \int_D \left(3\left(\phi^4 - \phi_{\text{FV}}^4\right) - 2v^2\left(\phi^2 - \phi_{\text{FV}}^2\right)\right) \simeq -\frac{3}{16\varepsilon\Delta^3}\,.$$

Running of the Euclidean action

$$S_R + S_{\text{ct}} = S\left(1 - \frac{9\lambda_0}{(4\pi)^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{\mu_0}\right)\right)$$

Non-trival check: the $\frac{1}{\varepsilon}$ pole and $\ln\mu$ cancel with the determinant

b) Functional determinant

We wish to get the eigenvalues of O and multiply them

The bounce and fluctuations are spherically symmetric, orbital decomposition

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-\frac{1}{2}} = \left|\prod_{l=0}^{\infty} \frac{\det' \mathcal{O}_l}{\det \mathcal{O}_{l\rm FV}}\right|^{-\frac{1}{2}}$$

$$\mathcal{O}_l = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{l\,(l+D-2)}{\rho^2} + V^{(2)}\,, \qquad V^{(2)} = \left.\frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2}\right|_{\bar\varphi}$$

Gel’fand-Yaglom theorem (appendix)

$$\mathcal{O}_l\,\psi_l = 0\,, \quad \psi_l(0) \sim \rho^l$$
$$\mathcal{O}_{l\rm FV}\,\psi_{l\rm FV} = 0\,, \quad \psi_{l\rm FV}(0) \sim \rho^l$$

$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\rm FV}} = \left(\left.\frac{\psi_l}{\psi_{l\rm FV}}\right|_{\infty}\right)^{d_l}$$

$$d_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!}$$ degeneracy $$d_0 = 1 \qquad d_1 = D$$

Fluctuations

Define the ratio $R_l \equiv \dfrac{\psi_l}{\psi_{l\mathrm{FV}}}$ and solve the stable equation

$$\ddot{R}_l + 2\left(\frac{\dot{\psi}_{l\mathrm{FV}}}{\psi_{l\mathrm{FV}}}\right)\dot{R}_l = \left(V^{(2)} - V^{(2)}_{\mathrm{FV}}\right) R_l$$

$$\psi_{l\mathrm{FV}}(0) \sim \rho^l\,, \qquad R_l(0) = 1\,, \qquad \dot{R}_l(0) = 0$$

IR $l \sim 1$ l UV $l \gg 1$

Low multipoles $l < \dfrac{1}{\Delta}$

Multiplicative TW expansion $\qquad R_l = \prod_{n\geq 0} R_{ln}^{\Delta^n} \qquad x = e^z$

$$R_{l0} = \frac{1}{(1+x)^2}\,,$$

$$\ln R_{l1} = 3\,(r + \ln x)\,,$$

$$\ln R_{l2}(x\to\infty) = \frac{3}{4}\frac{(l-1)\,(l+D-1)}{(D-1)^2}\,x^2$$

$$R_l(\infty) = \Delta^2 e^{D-1}\,\frac{3}{4}\,\frac{(l-1)\,(l+D-1)}{(D-1)^2}$$

Negative and zero modes ok

High multipoles $l \gg \frac{1}{\Delta}$

Shift multipoles $\nu = l + \frac{D}{2} - 1$

FV part $\psi_{\nu \mathrm{FV}} \simeq e^{k_\nu z}\,,$ $k_\nu^2 = 1 + \frac{\Delta^2 \nu^2}{r_0^2}$

Leading order $R_{\nu 0}(\infty) = \frac{(k_\nu - 1)\,(2k_\nu - 1)}{(k_\nu + 1)\,(2k_\nu + 1)}$

$$\ln R_\nu(\infty) = \ln \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right)$$

'All-order' corrections

Correct UV behaviour $\lim_{\nu \to \infty} R_\nu(\infty) = 1$ not low-l, though, as expected needs R_ν

Renormalized determinant

$$\sum_{\text{fin}} = \sum_{l} - \sum_{\text{asymp}} + \sum_{\text{ren}}$$

Two approaches (different power counting)

1a) organize in multipoles, minimal subtraction $\sum_{\text{asymp}} \sim \#_1 \nu + \#_2 \frac{1}{\nu}$

quadratic   log divergence

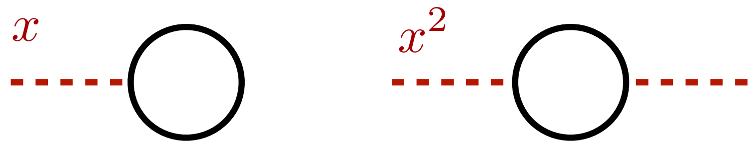
1b) organize in coupling x insertion $\sum_{\text{asymp}} \sim \#_1 x + \tilde{\#}_2 x^2$

Renormalize - replace divergencies with $\overline{\text{MS}}$ and introduce μ

2a) ζ function

$$\sum_{\text{ren}} \propto \frac{1}{\varepsilon} + \ln \mu + \text{fin}$$

2b) Feynman diagrams



Renormalized determinant

Sum over multipoles $$\ln\left(\frac{\det\mathcal{O}}{\det\mathcal{O}_{\rm FV}}\right)=\sum_{\nu=D/2-1}^{\infty} d_\nu \ln R_\nu\,, \qquad d_\nu\simeq\frac{2}{(D-2)!}\nu^{D-2}$$

Diverges in the UV as expected from QFT

$$\sum_{\nu\gg1} d_\nu \ln R_{\nu\gg1}\sim-\frac{3r_0(2-r_0)}{(D-2)!\Delta}\sum_{\nu\gg1}\nu^{D-2}\left(\frac{1}{\nu}-\frac{1}{\nu^3}\left(\frac{r_0}{2\Delta}\right)^2\right)$$

quadratic and log in *D=4*

Finite sum $$\Sigma_D=\sum_{\nu=\nu_0}^{\infty}\sigma_D=\sum_{\nu=\nu_0}^{\infty}d_\nu\left(\ln R_\nu-\ln R_\nu^a\right)$$

asymptotic subtraction

Renormalized determinants
(subtractions and logs for *Ds*)

WKB	Dunne, Min '05
ζ	Dunne, Kirsten '06
Improved	Hur, Min '08

Renormalized determinant *D=4*

$$\ln\left(\frac{\det\mathcal{O}}{\det\mathcal{O}_{\rm FV}}\right)=\sum_\nu \nu^2\left(\ln R_\nu-\frac{1}{2\nu}I_1+\frac{1}{8\nu^3}I_2\right)-\frac{1}{8}\tilde{I}_2$$

Asymptotic subtractions remove first two divergencies in any *D*

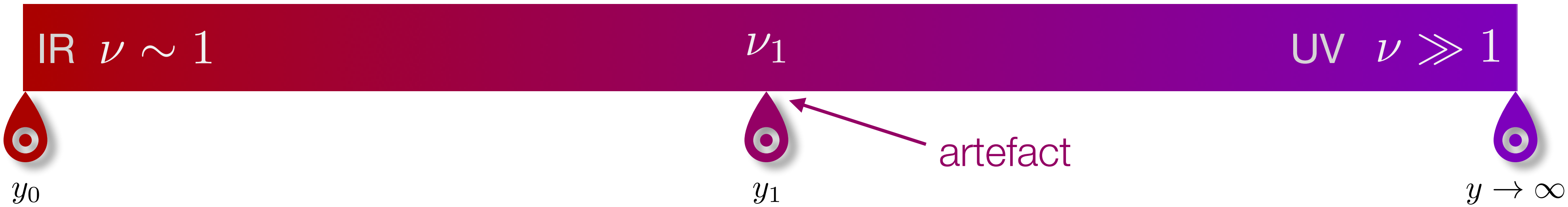
$$I_1=\int_0^\infty \mathrm{d}\rho\,\rho\left(V^{(2)}-V_{\rm FV}^{(2)}\right)\simeq-3\,(2-r_0)\left(\frac{r_0}{\Delta}\right),$$
$$I_2=\int_0^\infty \mathrm{d}\rho\,\rho^3\left(V^{(2)2}-V_{\rm FV}^{(2)2}\right)\simeq-3\,(2-r_0)\left(\frac{r_0}{\Delta}\right)^3$$

The renormalized piece gives the pole and the log scale part (and a finite piece, the 5/4)

$$\tilde{I}_2=\int_0^\infty \mathrm{d}\rho\,\rho^3\left(V^{(2)2}-V_{\rm FV}^{(2)2}\right)\left(\frac{1}{\varepsilon}+\gamma_E+1+\ln\left(\frac{\mu\rho}{2}\right)\right)$$
$$\simeq I_2\left(\frac{1}{\varepsilon}+\gamma_E+\frac{5}{4}+\ln\left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta}\right)\right)$$

Final sum done by Euler-Maclaurin, dominated by *large* multipoles, use $y = \dfrac{\Delta\nu}{r_0}$

$$\Sigma_4^{\int} \simeq \frac{1}{\Delta^3}\int_{y_0}^{\infty} \mathrm{d}y\, y^2 \left(\ln R_\nu + \frac{3}{2y} - \frac{3}{8y^3}\right) = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} + \ln 2y_0\right)$$



Separate low and high at arbitrary intermediate multipole ν_1

$$\Sigma_D = \Sigma_D^{\text{low}} + \Sigma_D^{\text{high}} = \sum_{\nu=\nu_0}^{\nu_1} \sigma_D + \sum_{\nu=\nu_1+1}^{\infty} \sigma_D$$

Combining low and high, we get the finite sum

$$\Sigma_4 = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta\right) \;\Rightarrow\; \ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}}\right) = \Sigma_4 - \frac{\tilde{I}_2}{8}$$

The γ_E and $\ln\Delta$ cancel with parts in $\tilde{I}_2 \simeq I_2\left(\frac{1}{\varepsilon} + \gamma_E + \frac{5}{4} + \ln\left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta}\right)\right)$

TW rate at one loop - summary

Ivanov, Matteini, MN, Ubaldi '22

Explicit closed form renormalized rate at one loop $\mu_0 = \sqrt{\lambda} v$

$$\frac{\Gamma}{\mathcal{V}} \simeq \left(\left(\frac{S}{2\pi} \right) \frac{12}{e^{D-1}} \lambda v^2 \right)^{D/2} \exp\left[-S - \frac{1}{\Delta^{D-1}} \begin{cases} \frac{20+9\ln 3}{54}, & D=3\,, \\ \frac{27-2\pi\sqrt{3}}{96}, & D=4\,, \end{cases} \right]$$

zero removal

determinant part

Renormalized action

$$S = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1\right) \Delta^2\right), & D=3 \\ \frac{\pi^2}{3\lambda} \left(1 - \left(2\pi^2 + \frac{9}{2}\right) \Delta^2\right), & D=4 \end{cases}$$

New & relevant + much more…

Matteini, MN, Shoji, Ubaldi '23

General procedure for even and odd *D*

Bounces and prefactors

are fascinating and relevant

Appendix

Zero removal

$$\Gamma \propto \frac{1}{\sqrt{\det \mathcal{O}'}} = \prod_n \sqrt{\frac{\lambda_{n\mathrm{FV}}}{\lambda'_n}} \propto (v^2)^{D/2}$$

Eigenvalues have *d*=2, drop *D* of them, sqrt overall

However, we're working with GY, so all the *l=1* eigenvalues are multiplied together

The trick: modified GY

$$\left(\mathcal{O}_{l=1} + \mu_\varepsilon^2\right) \psi_{l=1}^\varepsilon = 0$$

$$R_{l=1}^\varepsilon(\infty) = \frac{\psi_{l=1}^\varepsilon(\infty)}{\psi_{l=1}^{\mathrm{FV}}(\infty)} \simeq \frac{\left(\mu_\varepsilon^2 + \gamma_1\right) \prod_{n=2}^\infty \gamma_n}{\prod_{n=1}^\infty \gamma_n^{\mathrm{FV}}} = \mu_\varepsilon^2 R'_{l=1}(\infty)$$

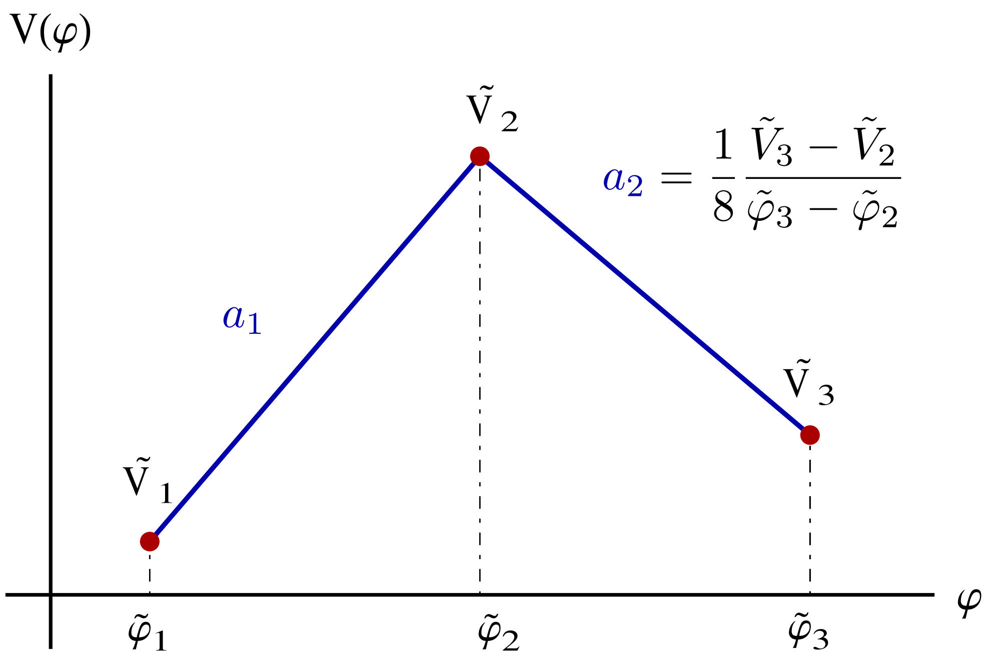
We already have the needed fluctuation functions, easy to off-set

$$R'_{l=1}(\infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2} R_{l=1}^\varepsilon(\infty) = \frac{e^{D-1}}{12} \frac{1}{\lambda v^2}$$

* two other ways of seeing the same thing give the same answer

This answers the question of dimensional analysis estimate

Triangular



Linear potentials

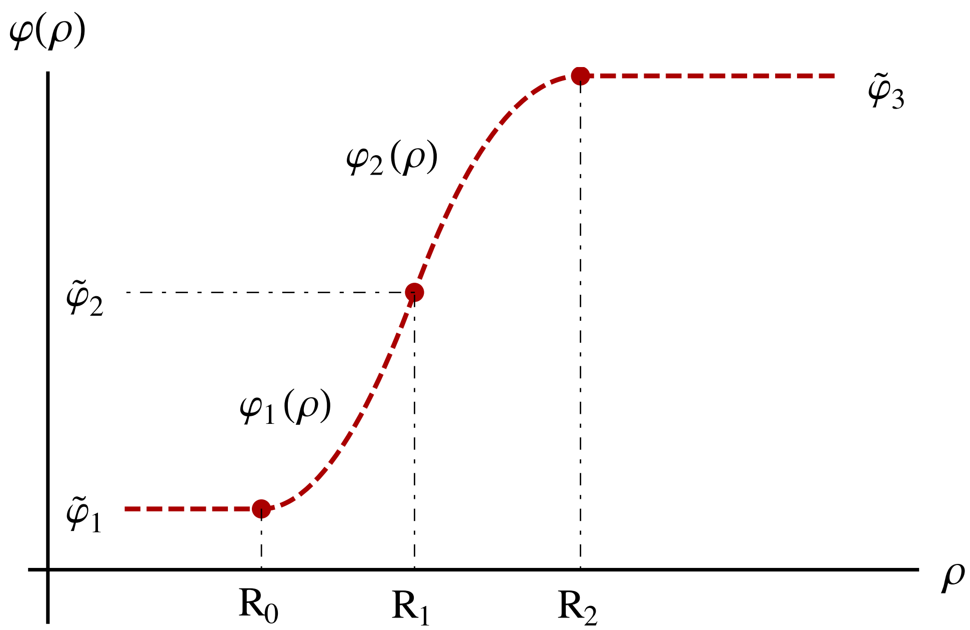
Duncan, Jensen '92

- triangle and box

Exact solution

$$\ddot\varphi + \frac{3}{\rho}\dot\varphi = dV = 8\,a$$

$$\varphi = v + a\,\rho^2 + \frac{b}{\rho^2}$$



Initial conditions @ R_0

- a) $\varphi_1(0) = \varphi_0, \quad \dot\varphi_1(0) = 0$

  $v_1 = \varphi_0, \quad b_1 = 0$

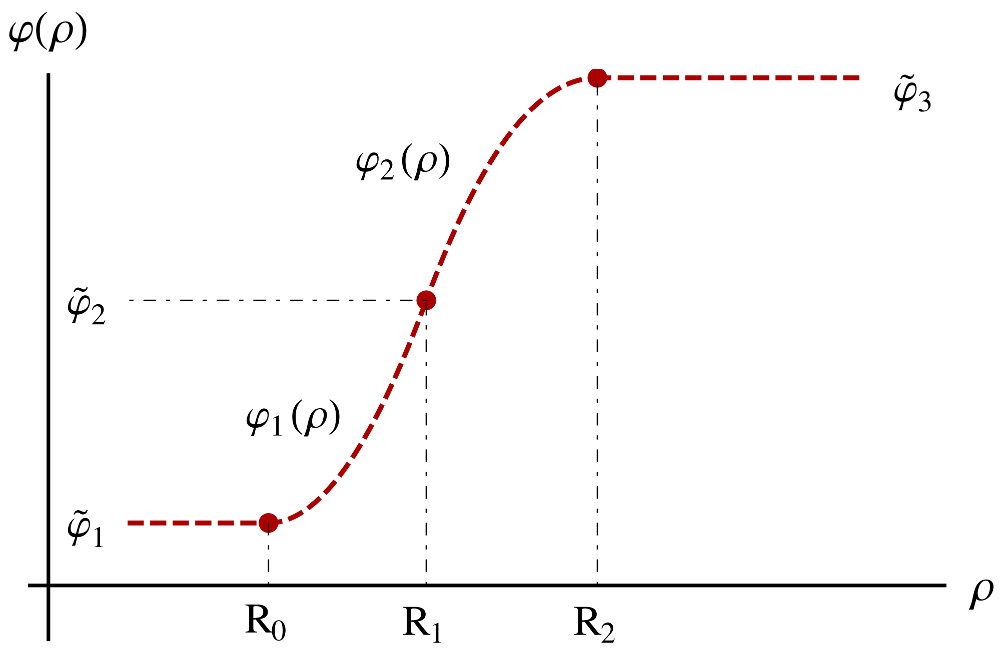
  shoot in φ_0

- b) $\varphi_1(R_0) = \tilde\varphi_1, \quad \dot\varphi_1(R_0) = 0$

  or R_0

  $v_1 = \tilde\varphi_1 - 2\,a_1\,R_0^2, \quad b_1 = a_1\,R_0^4$

Triangular



Matching conditions @ R_1

$$\varphi_1(R_1) = \varphi_2(R_1) = \tilde{\varphi}_2, \quad \dot{\varphi}_1(R_1) = \dot{\varphi}_2(R_1)$$

Final conditions @ R_2

$$\varphi_2(R_2) = \tilde{\varphi}_3, \quad \dot{\varphi}_2(R_2) = 0$$

$$v_2 = \tilde{\varphi}_3 - 2\, a_2\, R_2^2, \quad b_2 = a_2\, R_2^4$$

Complete solution - a) works in D-dimensions

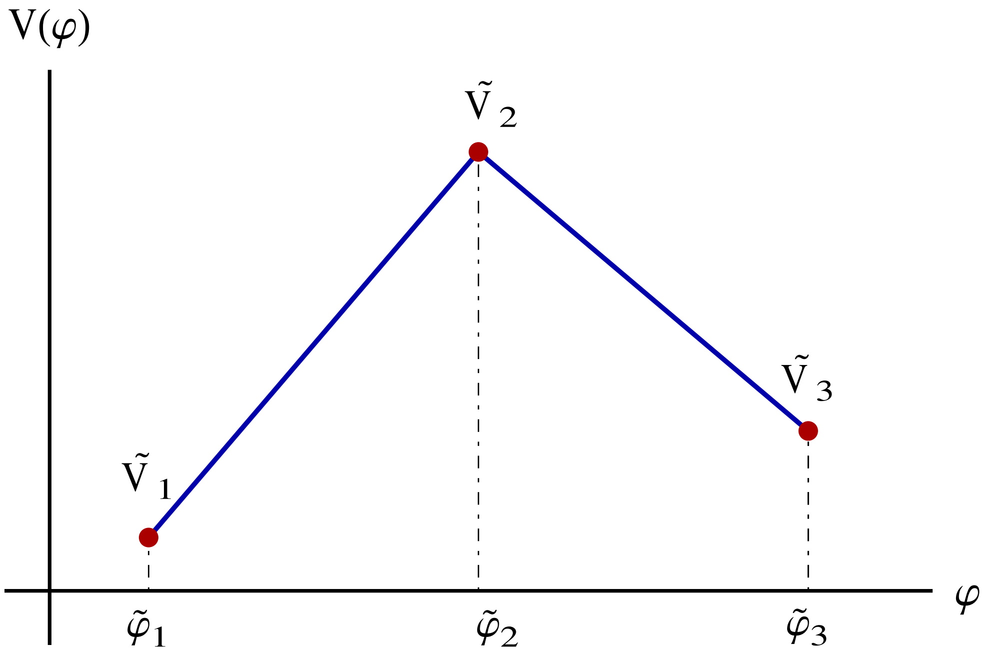
- a) $$\varphi_0 = \frac{\tilde{\varphi}_3 + c\,\tilde{\varphi}_2}{1+c}, \qquad c = 2\frac{a_2 - a_1}{a_1}\left(1 - \sqrt{\frac{a_2}{a_2 - a_1}}\right) \qquad R_1 = \sqrt{\frac{D}{4}\left(\frac{\tilde{\varphi}_2 - \varphi_0}{a_1}\right)}$$

- b) $$R_1 = \frac{1}{2}\frac{\tilde{\varphi}_3 - \tilde{\varphi}_1}{\sqrt{a_1\,(\tilde{\varphi}_2 - \tilde{\varphi}_1)} - \sqrt{-a_2\,(\tilde{\varphi}_3 - \tilde{\varphi}_2)}}$$

$$R_0^2 = R_1\left(R_1 - \sqrt{\frac{\tilde{\varphi}_2 - \tilde{\varphi}_1}{a_1}}\right)$$

$$R_2^2 = R_1\left(R_1 + \sqrt{\frac{\tilde{\varphi}_3 - \tilde{\varphi}_2}{-a_2}}\right)$$

Summary



Complete exact analytic solution

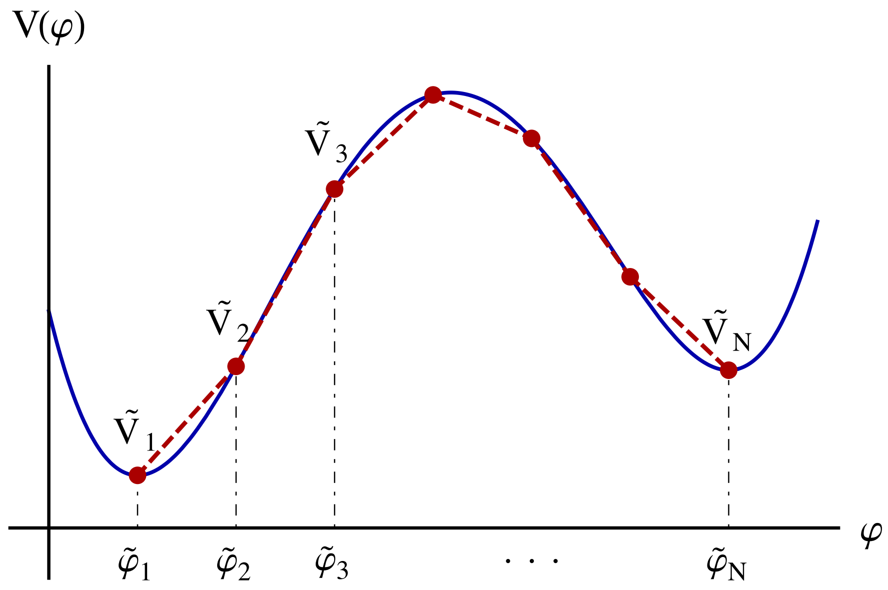
Solved in terms of Euclidean radius

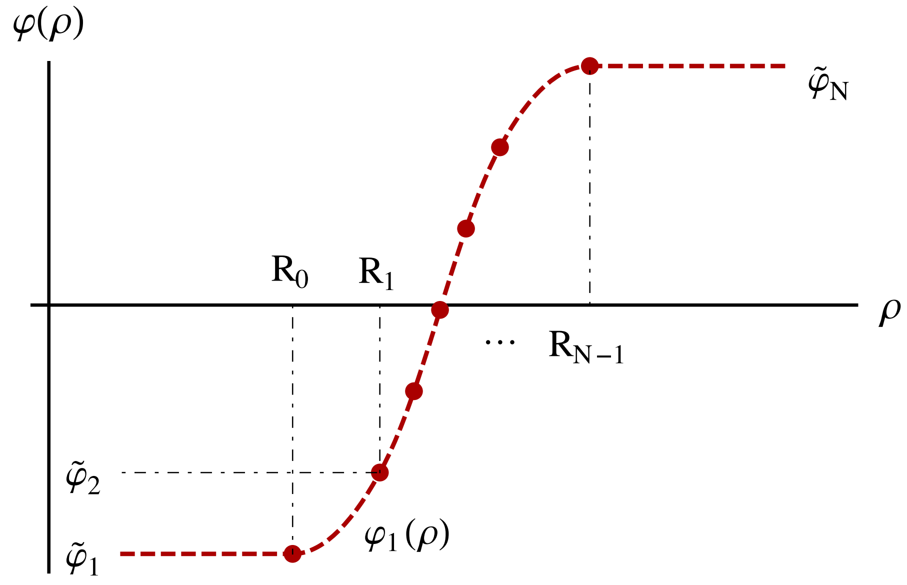
Stable in thin wall, goes over to TWA with limited validity

Dutta, Hector, Konstandin, Vaudrevange, Westphal '12

Analytic continuation in Minkowski space describes the bubble evolution Pastras '11

Polygonal bounces





$$V_i(\varphi) = \underbrace{\left(\frac{\tilde{V}_{i+1} - \tilde{V}_i}{\tilde{\varphi}_{i+1} - \tilde{\varphi}_i}\right)}_{8\,a_i} (\varphi - \tilde{\varphi}_i) + \tilde{V}_i - \tilde{V}_N, \qquad dV_i = 8\,a_i.$$

No free parameters, one segment three unknowns $v_i,\ b_i,\ R_i$

Generalize case b), solve R_0 or R_i a), retrieve φ_0

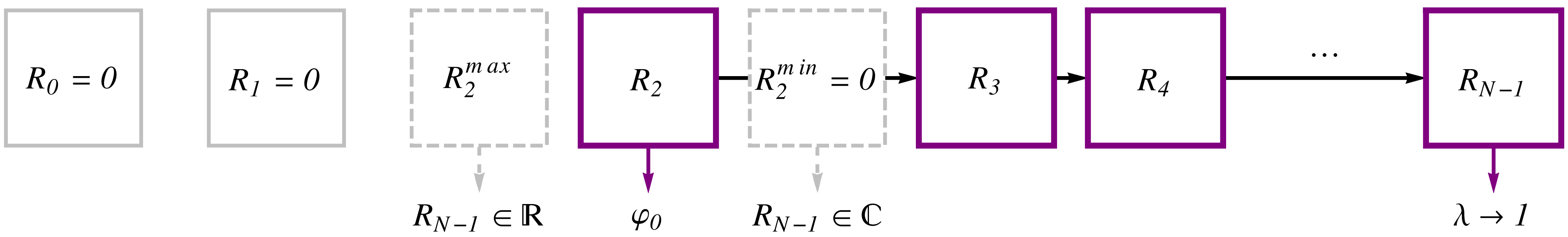
Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

fewnomial

$$R_n^D - \frac{D}{4}\frac{\delta_n}{a_n}R_n^{D-2} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0$$

$$\delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots



Bounce action

Euclidean action

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1}\, \mathrm{d}\rho \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

PB action

$$S_{>2} = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{R_0^D}{D}\left(\tilde{V}_1 - \tilde{V}_N\right) + \sum_{i=1}^{N-1} \left[\rho^2 \left(\frac{32 a_i^2 (D+1)\rho^D}{D^2(D+2)} + \frac{16 a_i b_i}{D(D-2)} - \frac{2b_i^2}{\rho^D (D-2)}\right) + \frac{\rho^D}{D}\left(8a_i\left(v_i - \tilde{\varphi}_i\right) + \tilde{V}_i - \tilde{V}_N\right)\right]_{R_{i-1}}^{R_i} \right\}$$

Total

$$S = \mathcal{T} + \mathcal{V}$$

$$\mathcal{T} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho\, \dot{\varphi}^2, \qquad \mathcal{V} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho\, \mathrm{V}(\varphi)$$

kinetic potential

Derrick's theorem

Non-existence of non-trivial static solutions of KG equation, no solitonic scalar 'particles'

Derrick '64

Unstable under re-scaling $\varphi(\rho) \to \varphi(\rho/\lambda)$

$$\lambda \times 0 = 0,$$
$$\lambda \times \infty = \infty$$

$$S_D^{(\lambda)} = \lambda^{D-2}\mathcal{T} + \lambda^D \mathcal{V}$$

change of variables...remain the same

action is extremized at non-scaled values for true solutions

$$\left.\frac{dS_D^{(\lambda)}}{d\lambda}\right|_{\lambda=1} = 0$$

$$(D-2)\mathcal{T} + D\mathcal{V} = 0$$

relation between kinetic and potential

$$\left.\frac{d^2 S_D^{(\lambda)}}{d\lambda^2}\right|_{\lambda=1} < 0$$

Caveat for PB $R \to \lambda R$

Works for $N \gg 1$

Benchmarks

Back to thin wall

Coleman '77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right)$$

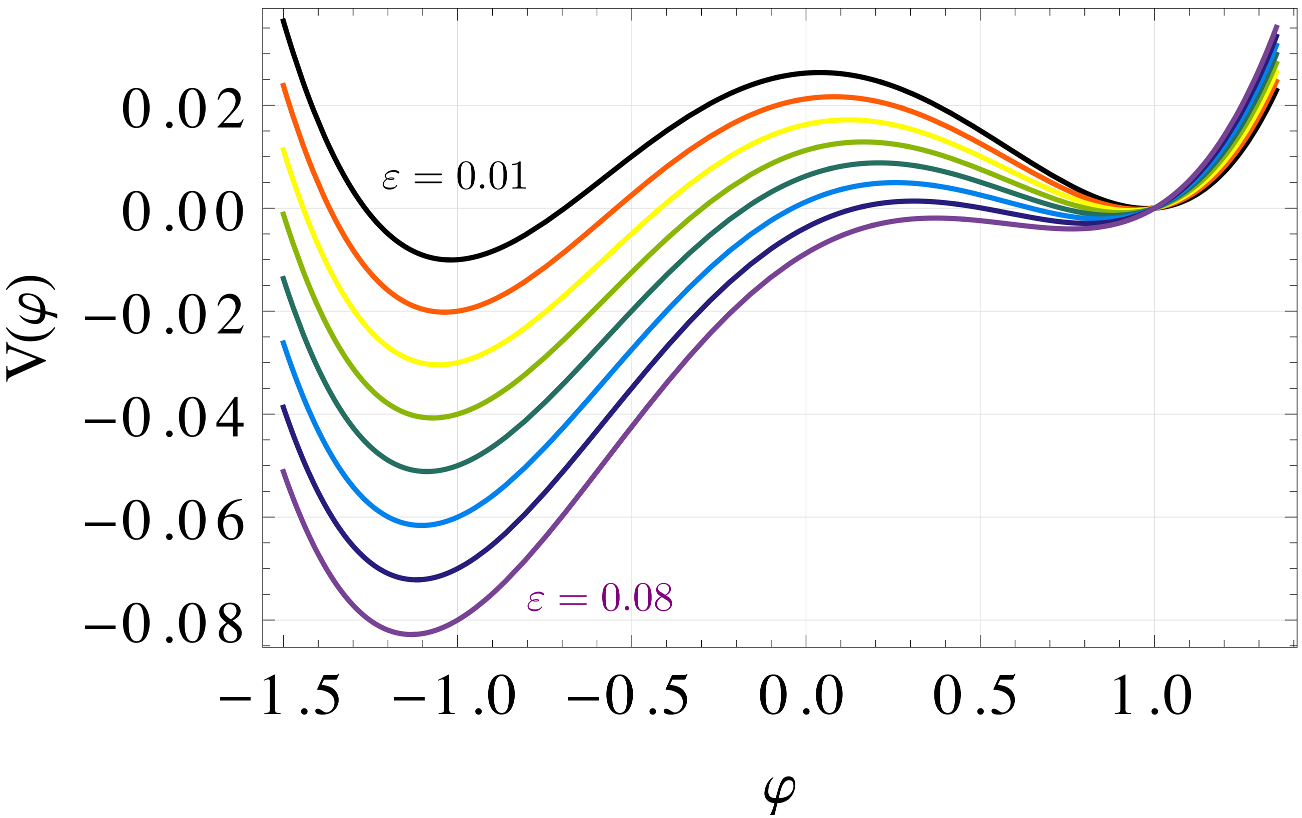
Benchmark for testing $\lambda = 0.25, \quad v = 1$

rescaling

Sarid '98

$\varepsilon \to 0$

thin wall



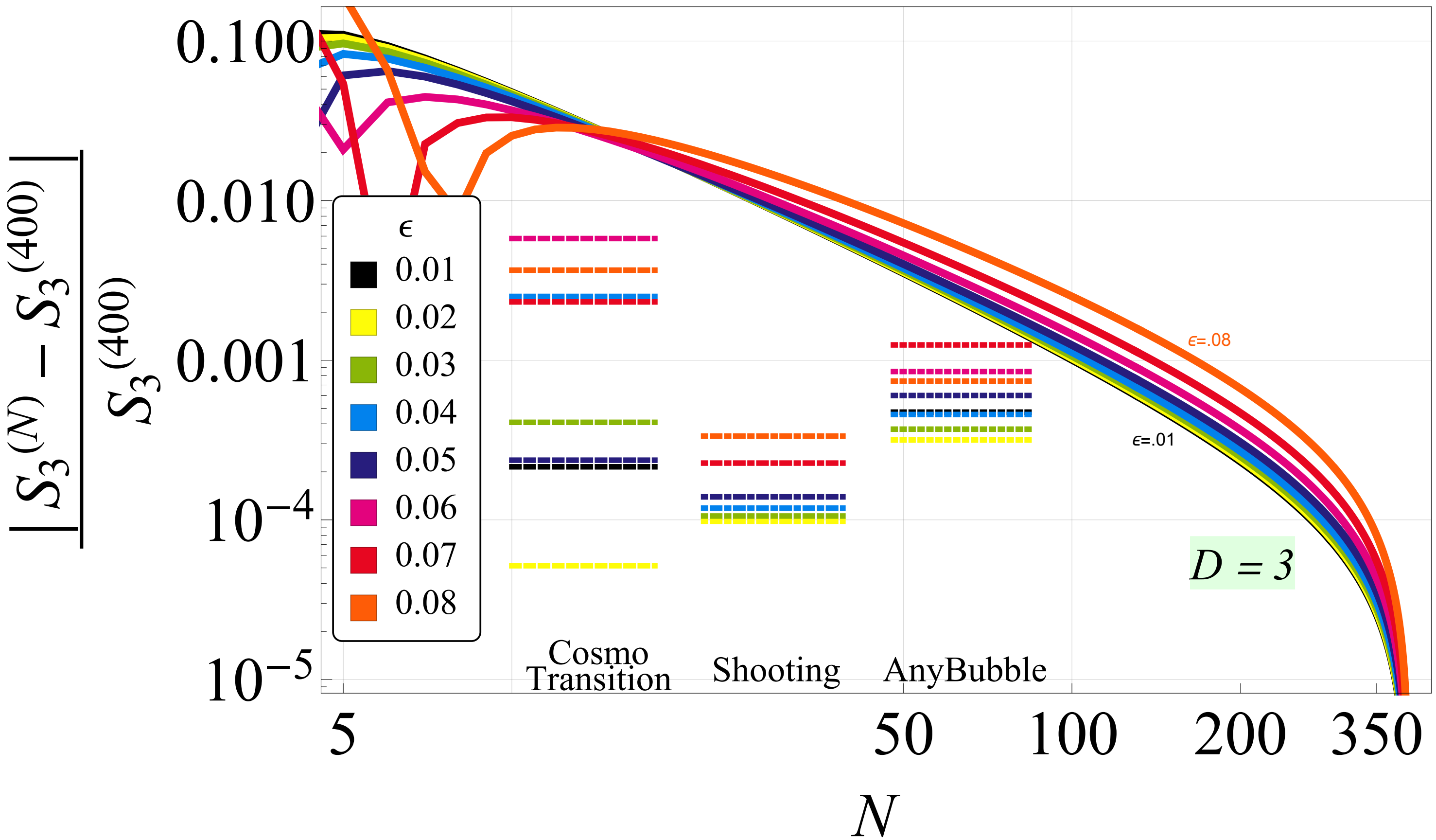
thick wall

$\varepsilon > 0.8$

2nd minimum disappears

Euclidean action, comparisons

- CosmoTransitions — Runge-Kutta PDE solver, initial value approximations — **Wainwright '11**
  discontinued
- AnyBubble — multiple shooting, damping approximations — **Masoumi, Olum, Shlaer '16**
- Shooting — Mathematica, precise setting of initial values, issues with 0, infinity



PB within permille after 100 iterations

stable for thin/thick

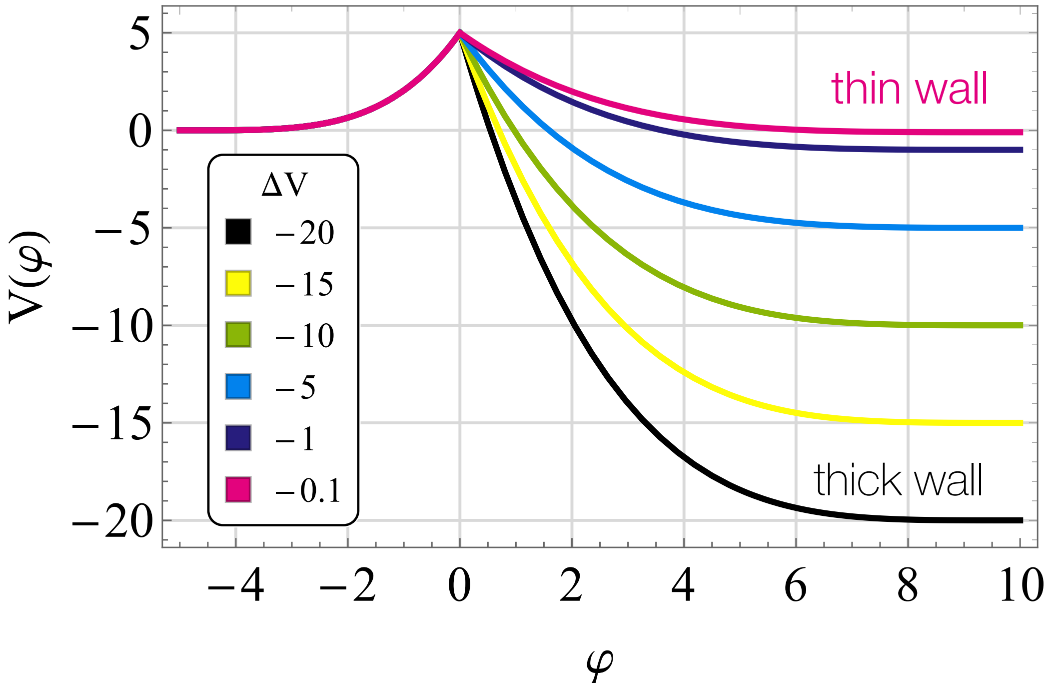
faster convergence for thin wall

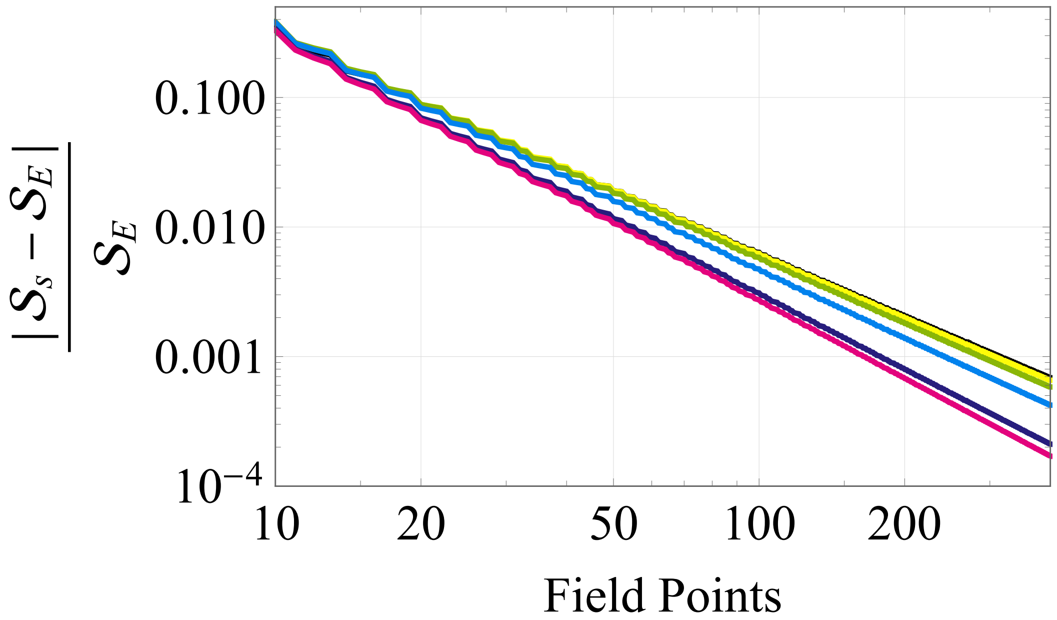
Bi-quartic

Other exact N=3 potentials, quartic-linear, quartic-quartic

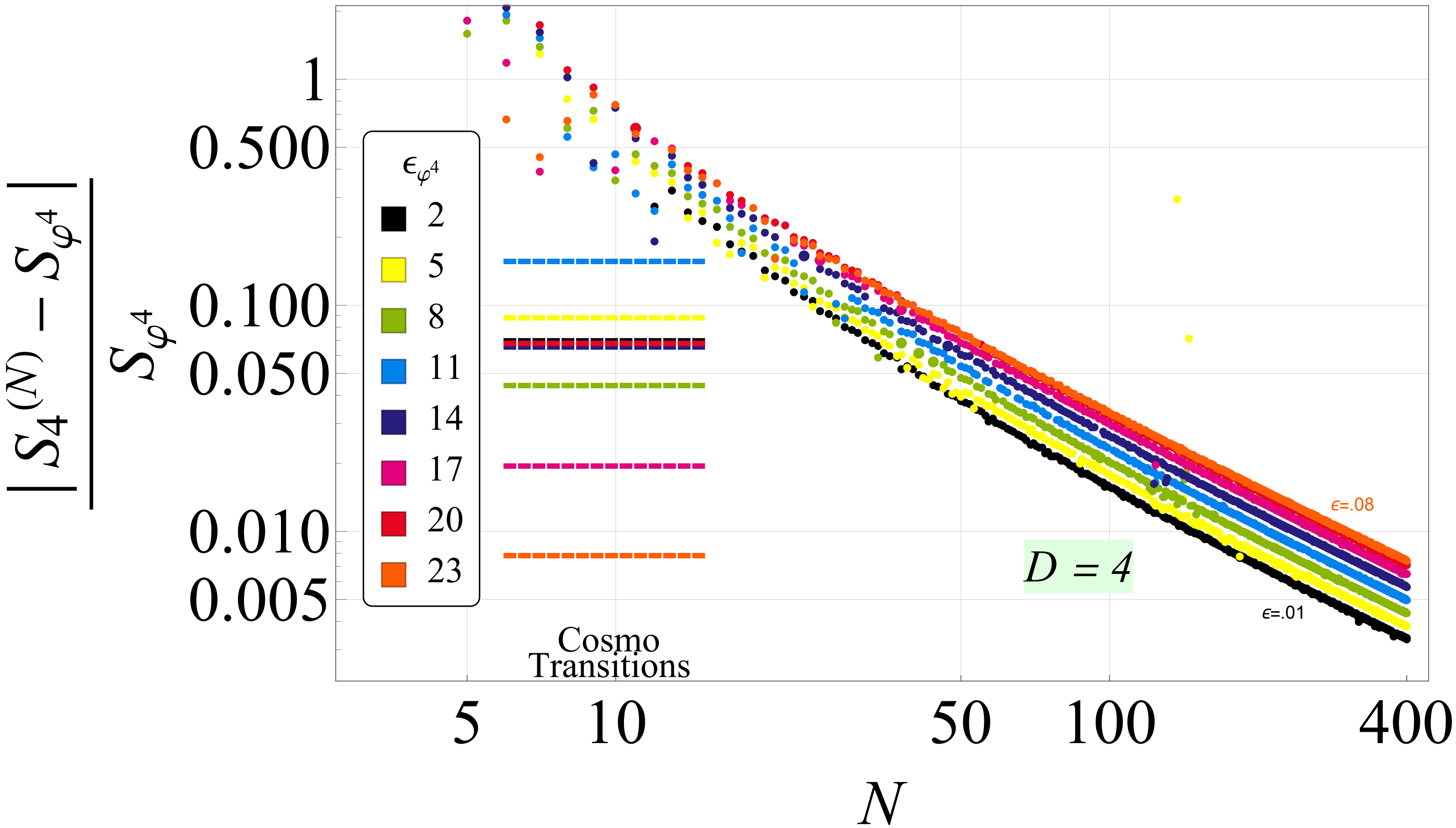
Dutta, Hector, Vaudrevange, Westphal '11

known exact solution, 'fair' comparison and test for the PB method





- **CosmoTransitions** — fails with the action, possible to repair by hand, precise from 20% to 0.5%
- **AnyBubble** — fails to compute
- **Polygonal bounce** — works smoothly with a bi-homogeneous segmentation

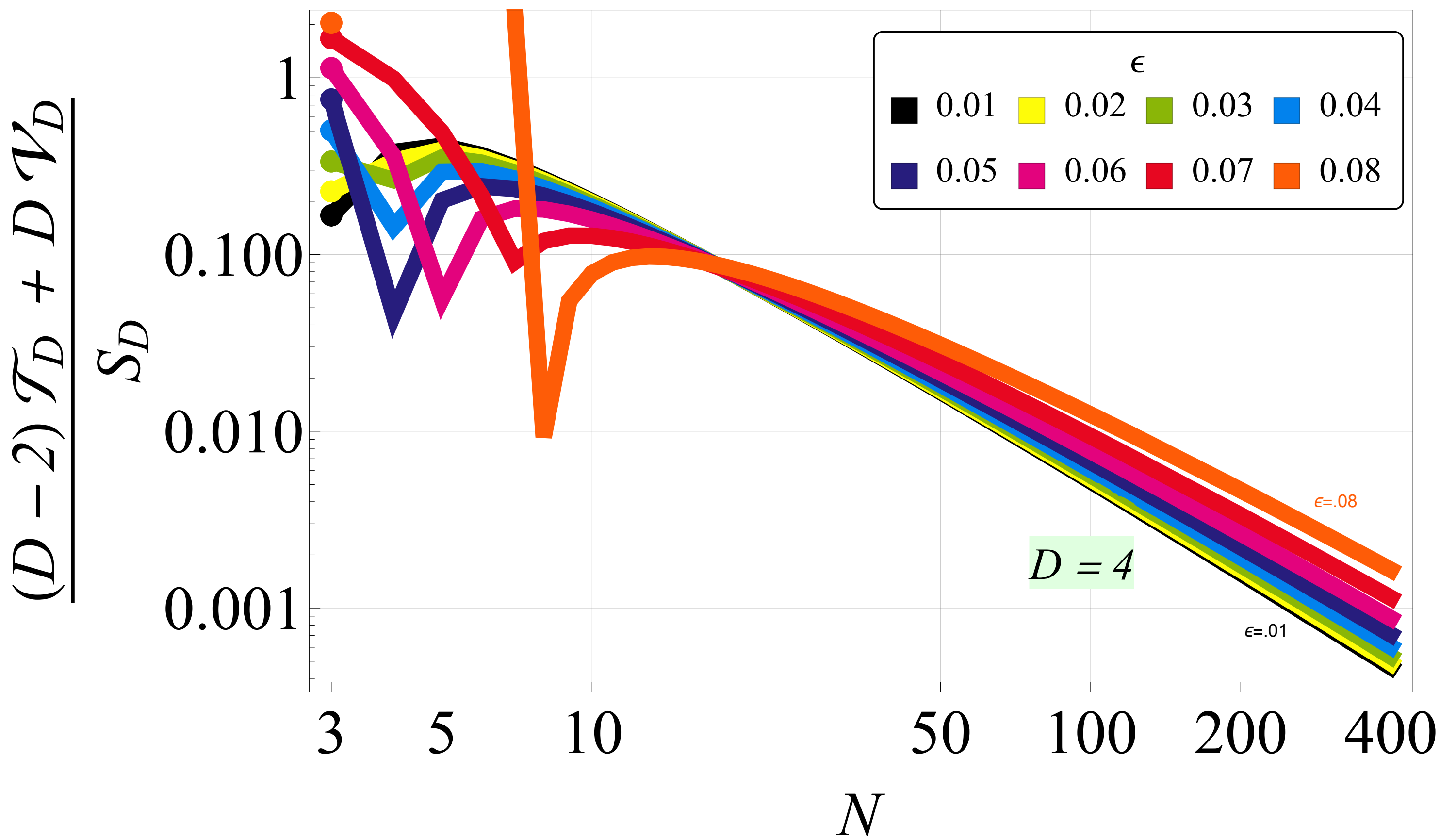


Derrick's theorem $(D-2)\mathcal{T} + D\mathcal{V} \to 0$

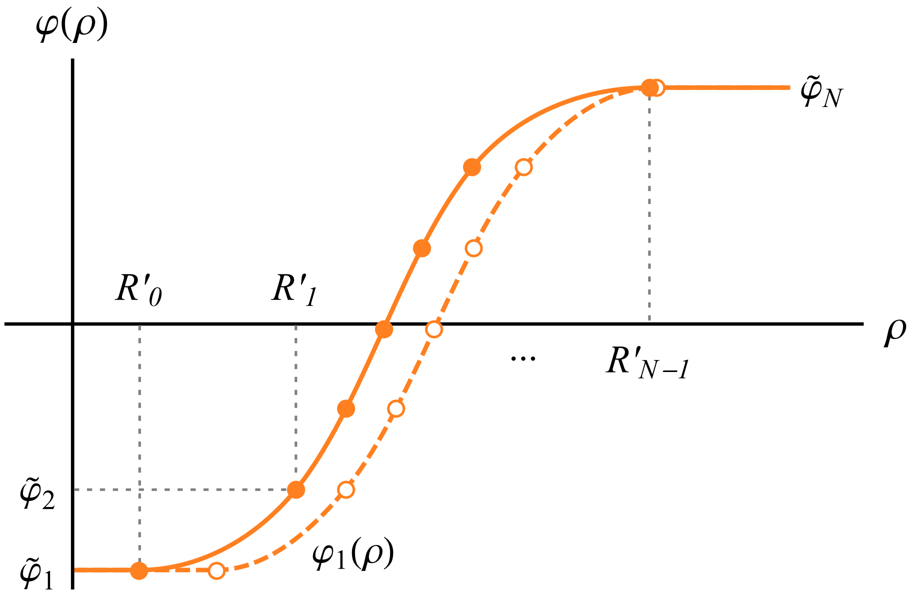
finite part corrections up to $N \simeq 10$

independent measure of goodness of approximation

above relation 'exact' for the PB potential



Higher orders



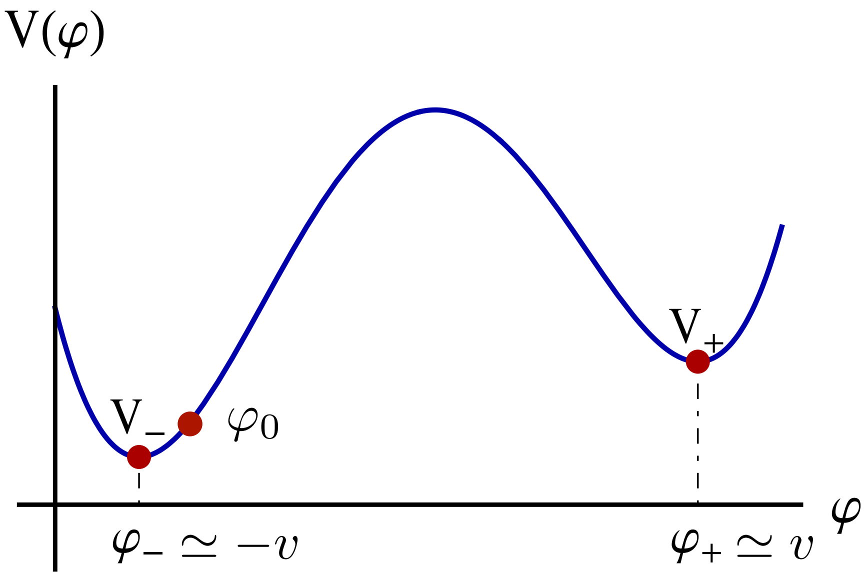
Match at perturbed radii

$$R_s \to R_s\,(1 + r_s)\,, \qquad r_s \ll 1$$

Rederive the matching conditions

A single linear equation = very fast

$$r_s = \frac{\beta_s + \frac{D-2}{2}\left(\nu_s + \mathcal{I}_s + \frac{4}{D}\alpha_s R_s^2\right) R_s^{D-2}}{(D-2)\left(b_s - \frac{4}{D} a_s R_s^D\right)}$$



Thin wall approximation

Coleman '77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right), \qquad S_1 = \frac{v^3\sqrt{\lambda}}{3}$$

small ε limit $\quad \varphi_0 \simeq \varphi_-$ $\quad$ until $\quad \rho = R$

field solution

$$\varphi(\rho) = \begin{cases} -v, & \rho \ll R \\ \varphi_1(\rho - R), & \rho \approx R \\ v, & \rho \gg R \end{cases} \qquad \varphi_1(\rho) = v\tanh\left(\frac{\sqrt{\lambda}v}{2}\rho\right)$$

bounce action

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left(\frac{1}{2}\dot{\varphi}^2 + V\right)$$

$$= -\frac{1}{2}\pi^2 R^4 \varepsilon + \pi^2 R^3 S_1$$

volume $\qquad$ surface

extremize the action

$$\frac{dS_E}{dR} = 0 \quad \Rightarrow \quad R = \frac{3S_1}{\varepsilon}$$

$$S_E = \frac{27\pi^2}{2}\frac{S_1^4}{\varepsilon^3}$$

$$\frac{d^2 S_E}{dR^2} < 0$$

runaway

Coleman '77

Bödeker, Moore '09, '17

Custom options control the input

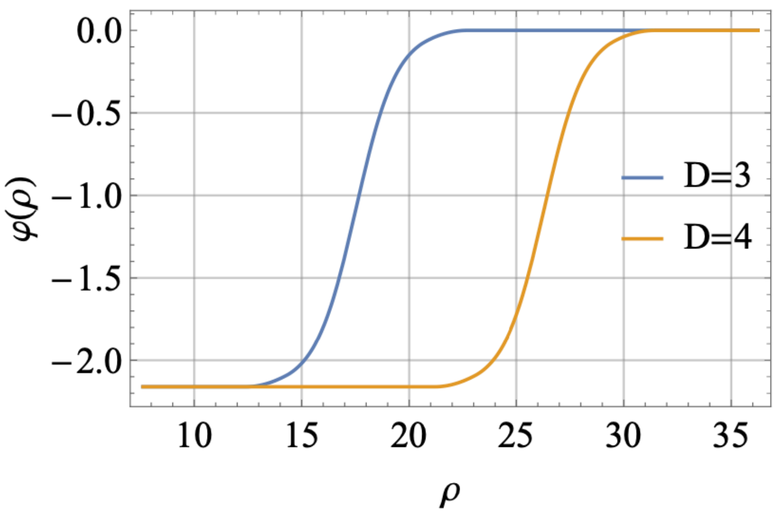
`“Dimension”`	sets the Eucludiean spacetime dimension, 3 or 4
`“FieldPoints”`	number of field points defines the segmentation
`“Gradient”`	one can pre-calculate the gradient, or make it numerical
`“Hessian”`	similar to the gradient, needed for multi-fields
`“MaxPath”`	limits the number of path iterations, typically small
`“MidFieldPoint”`	one can define a starting fixed point (e.g. the saddle)
`“PathTolerance” &` `“ActionTolerance”`	set a goal for the precision of the path variation and the Euclidean action

Output is a bundled container that can be easily accessed

`"Action"`	sets the Eucludiean spacetime dimension, 3 or 4
`"Bounce"`	number of field points defines the segmentation
`"Coefficients"`	one can pre-calculate the gradient, or make it numerical
`"Path"`	similar to the gradient, needed for multi-fields
`"Radii"`	limits the number of path iterations, typically small

In[9]:=
```
BouncePlot[{bf3,bf}, PlotLegends-> Placed[{"D=3","D=4"}, {Right,Center}]]
```

Out[9]=

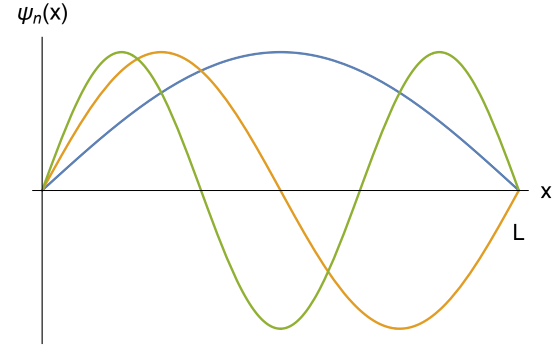


custom function for convenient plotting

Gel'fand-Yaglom aside

simplest instance of GY formalism 'magic'

QM well, classical 1D string



$$\mathcal{O} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2 \qquad \mathcal{O}_{\mathrm{FV}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

a) impose Dirichlet (fixed) boundary condition at L

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 + m^2\,, \quad \lambda_{n\mathrm{FV}} = \left(\frac{n\pi}{L}\right)^2\,, \quad \frac{\det\, \mathcal{O}}{\det\, \mathcal{O}_{\mathrm{FV}}} = \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_{n\mathrm{FV}}} = \frac{\sinh(mL)}{mL}$$

b) solve the Cauchy (open) boundary condition

$$\psi'' - m^2\psi = 0, \quad \psi(0) = 0, \quad \psi'(0) = 1, \quad \psi = \frac{\sinh(mx)}{m}, \quad \psi_{\mathrm{FV}} = x$$

$$\frac{\det\, \mathcal{O}}{\det\, \mathcal{O}_{\mathrm{FV}}} = \frac{\psi}{\psi_{\mathrm{FV}}}(L) = \frac{\sinh(mL)}{mL}$$

Classical physical significance?