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Lorentzian approach to the path integral

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Takumi Hayashi, Kohei Kamada, **N.O.**, Jun’ichi Yokoyama, arXiv: 2112.09284

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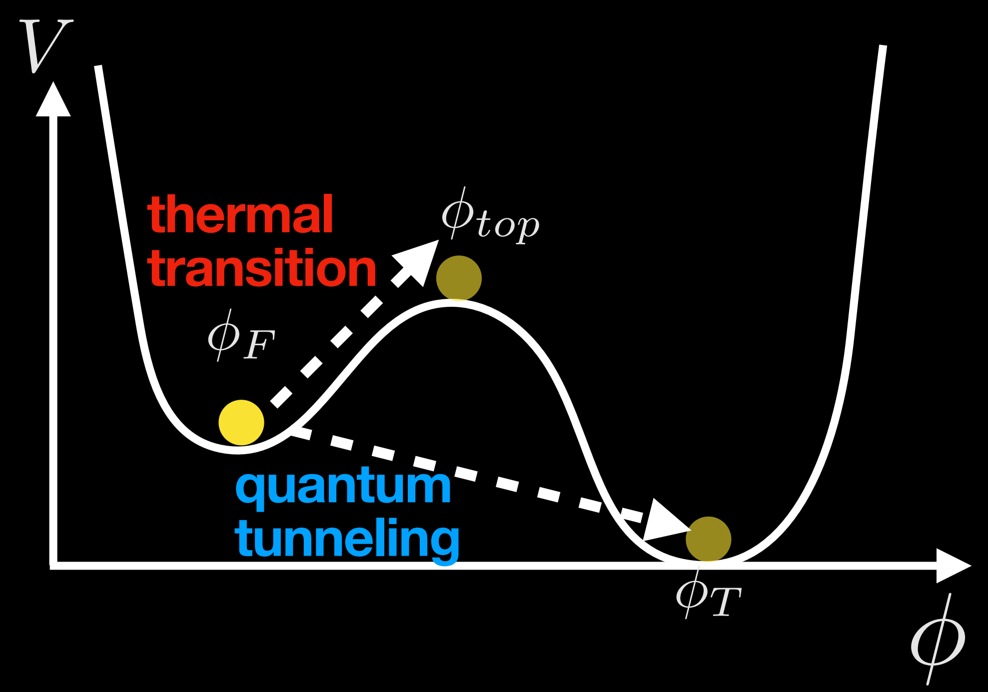
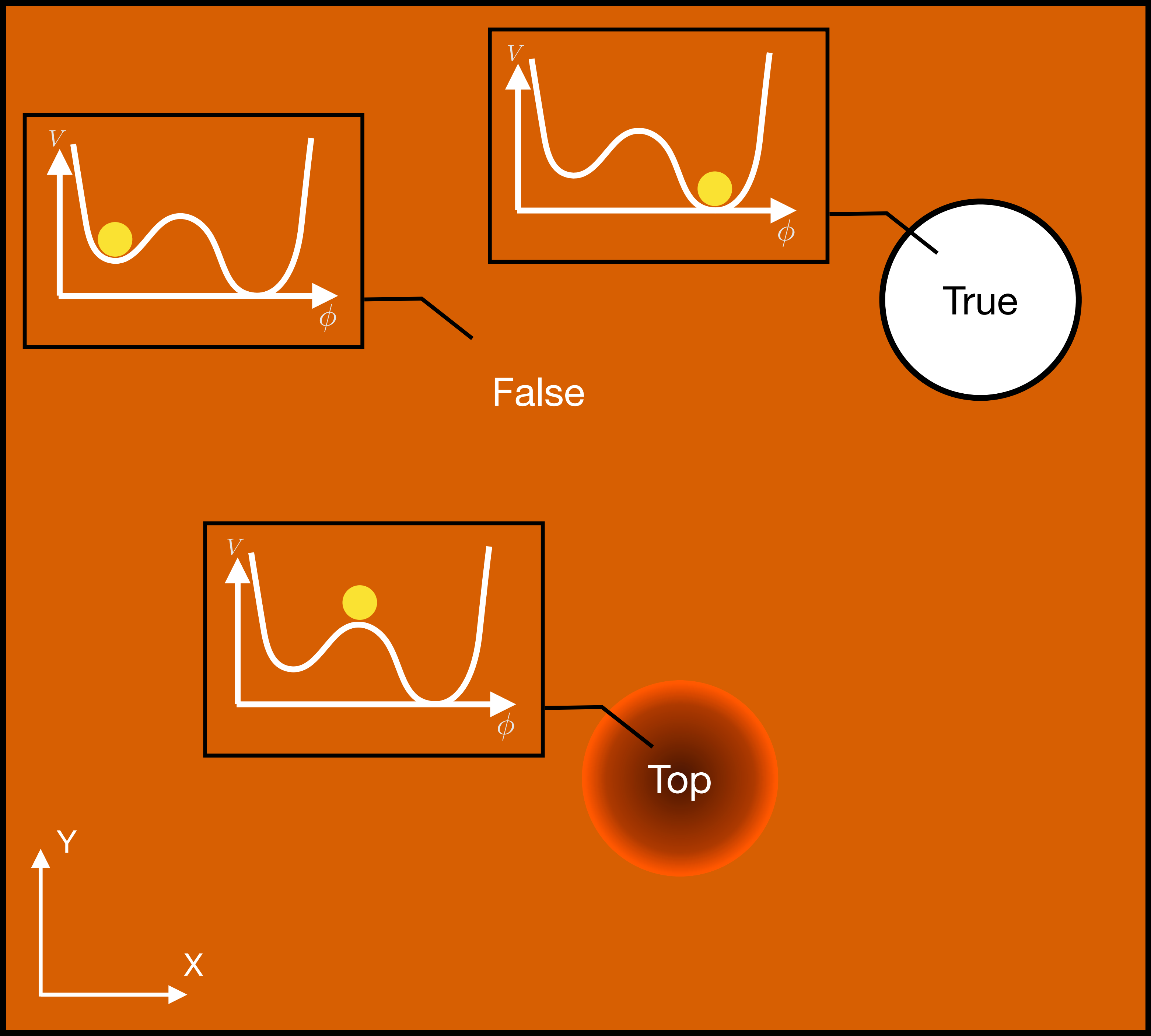
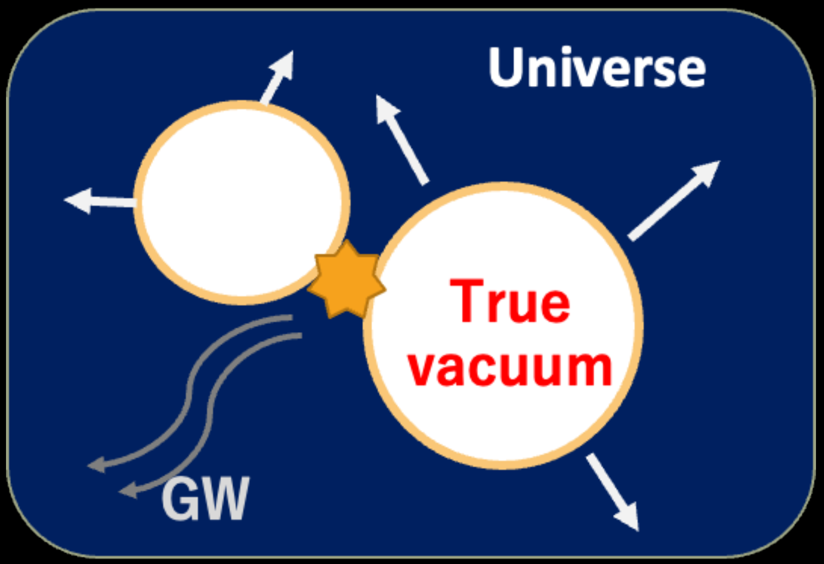
Vacuum phase transition

quantum tunneling or **thermal transition** of matter fields





Vacuum bubbles



picture: T. Hayashi

- GW emission caused by bubble collisions
- Higgs metastability in (thermal) early Universe
- Vacuum decay seeded by black holes → constraints on PBH parameters or the parameters of Higgs potential

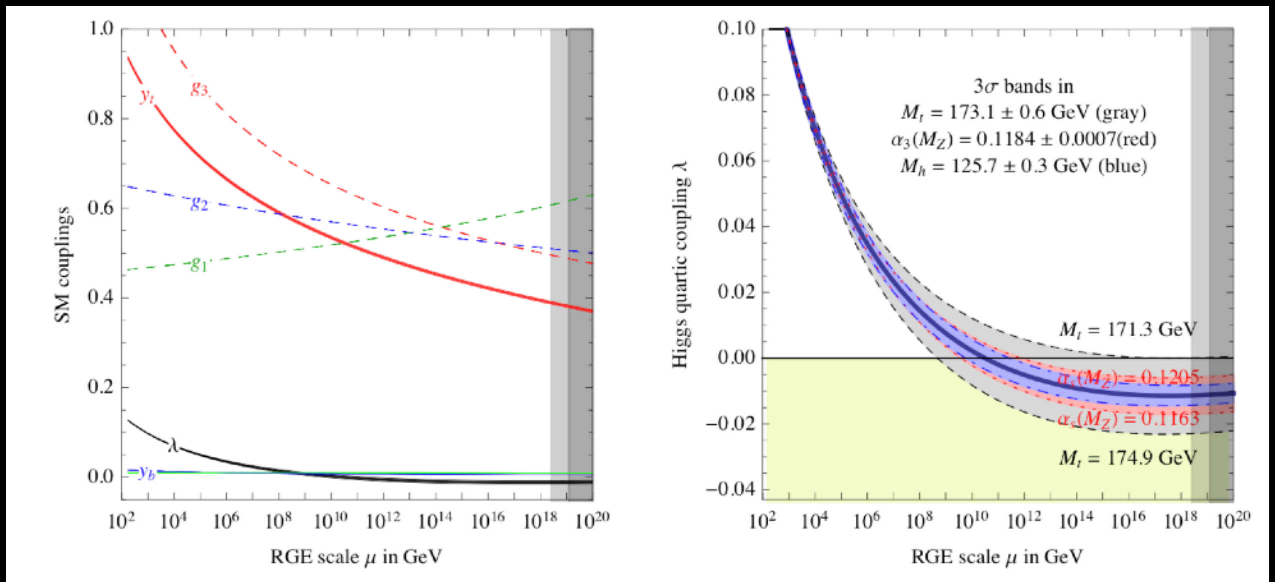
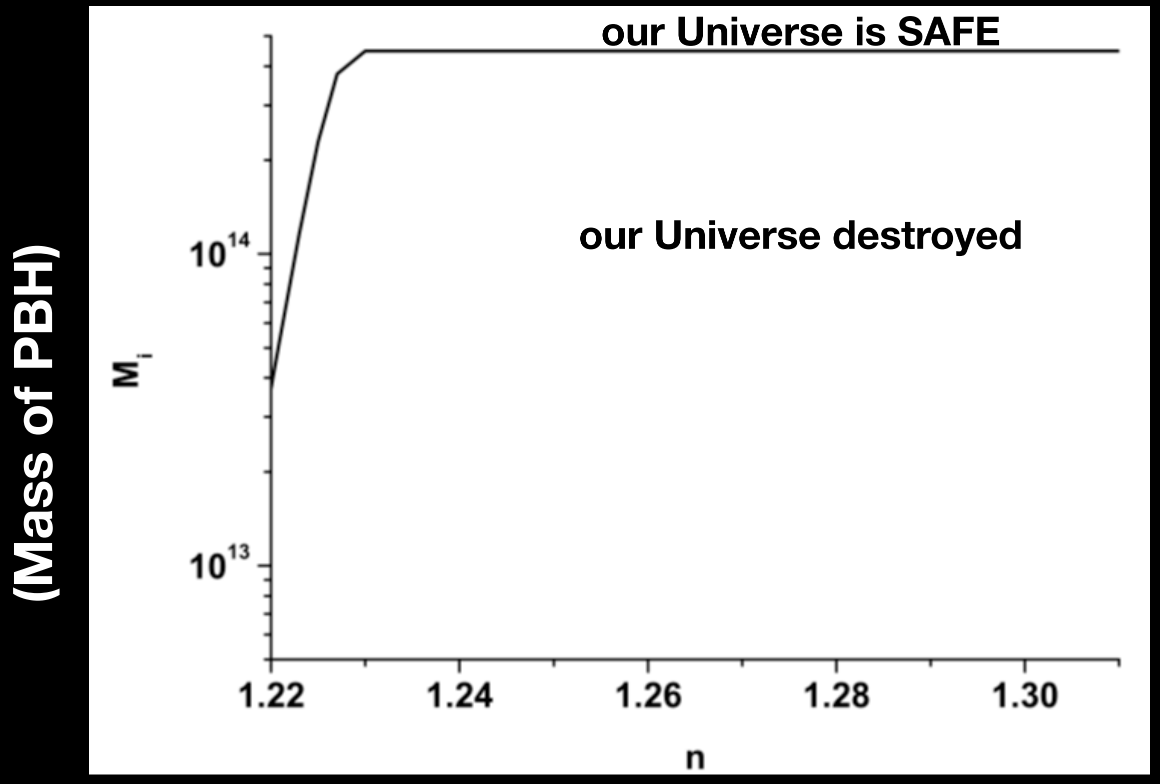


Figure 1: **Left**: *SM RG evolution of the gauge couplings* $g_1 = \sqrt{5/3}g'$, $g_2 = g$, $g_3 = g_s$, *of the top and bottom Yukawa couplings* (y_t, y_b), *and of the Higgs quartic coupling* λ. *All couplings are defined in the* $\overline{\text{MS}}$ *scheme. The thickness indicates the* $\pm 1\sigma$ *uncertainty.* **Right**: *RG evolution of* λ *varying* M_t, M_h *and* α_s *by* $\pm 3\sigma$.

Degrassi et al. (2013)



(spectral index) Dai+ (2019)

Euclidean v.s. Lorentzian

- Lorentzian path integral is oscillatory. NOT absolutely convergent.
- Euclidean path integral enable us to have a finite result in the amplitude.
- Can the Euclidean path integral be applicable to a general metric?
- In the existence of gravity, there is the “conformal factor problem”.
- Euclidean path integral is easy to use. (Can be wrong.)
- Lorentzian path integral is technical but important. (May be true.)

Lorentzian path integral?
-application to quantum cosmology-

What's the probability of the birth of the de Sitter universe out of "nothing"?

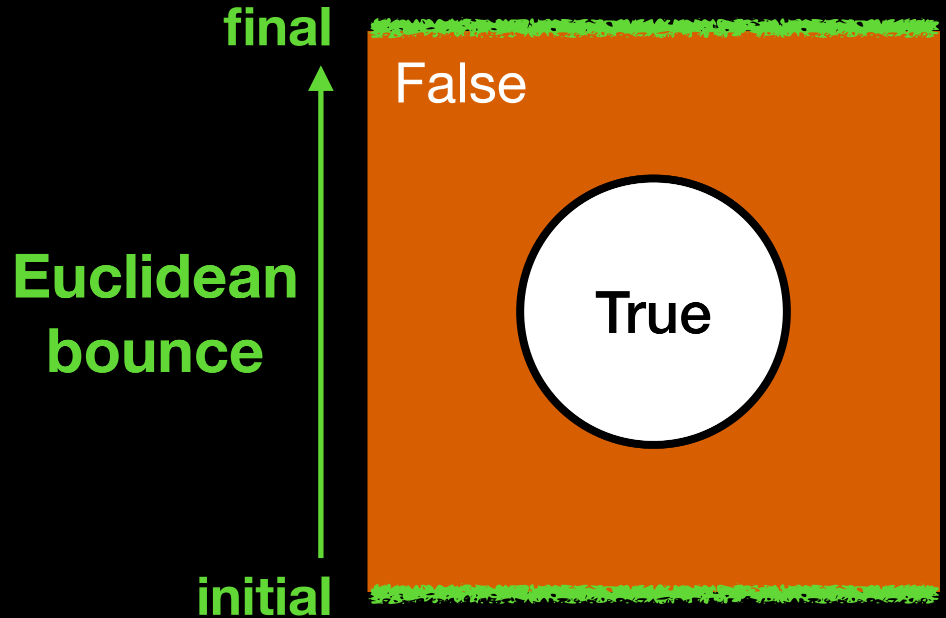
Λ : cosmological const.

Hartle and Hawking $t \to -i\tau$ Hawking (1982); Hartle (1983)	$\Gamma \sim e^{+12\pi^2/\Lambda}$	Euclidean path integral
Vilenkin Vilenkin (1982)	$\Gamma \sim e^{-12\pi^2/\Lambda}$	Wheeler-DeWitt+ WKB approximation
Linde $t \to +i\tau$ Linde (1984)	$\Gamma \sim e^{-12\pi^2/\Lambda}$	Euclidean path integral + inverse Wick rotation
Feldbrugge, Lehners, Turok Feldbrugge, Lehners, Turok(2017)	$\Gamma \sim e^{-12\pi^2/\Lambda}$	Lorentzian path integral

Let’s apply the Lorentzian path integral to Vacuum decay process.

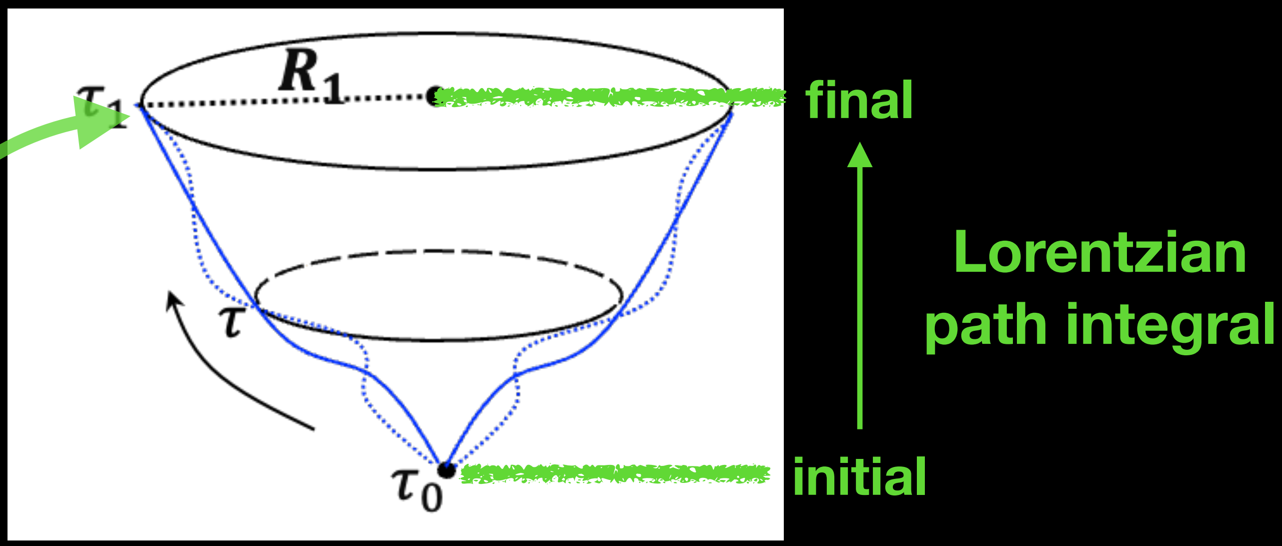
Standard

Ours



Coleman (1977)

Callan and Coleman (1977)

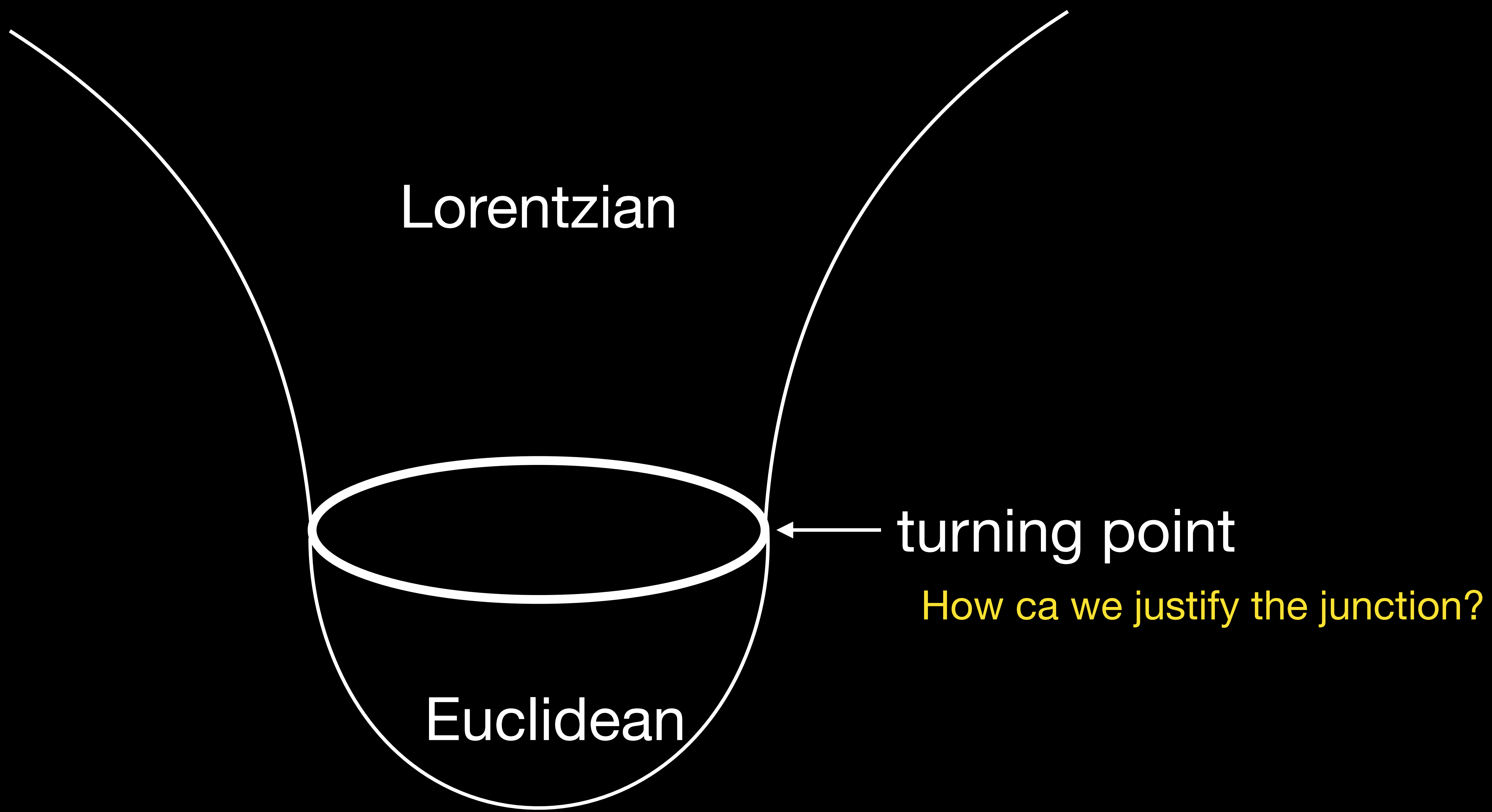


picture: T. Hayashi

Set a bubble of arbitrary size

as the final state

Standard formulation of vacuum bubbles



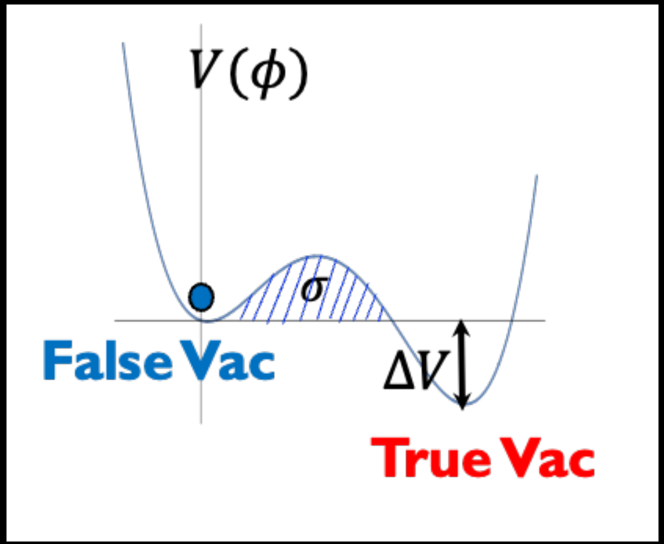
Setup

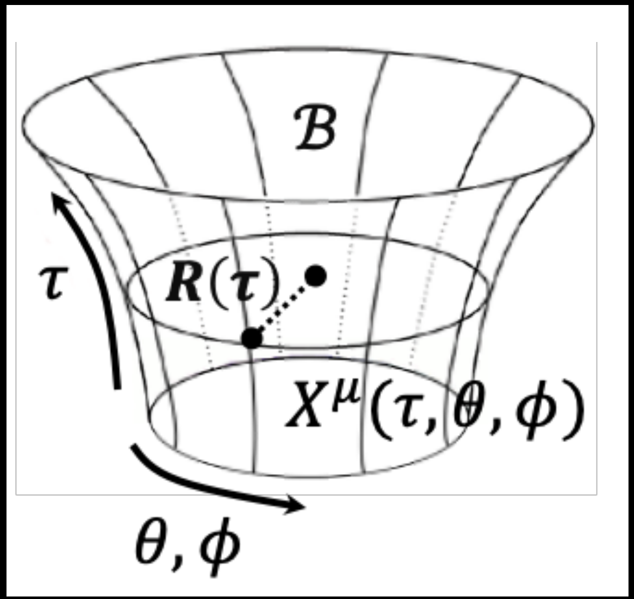
1. Assumption (for simplification)

- thin wall
  details of a potential barrier → energy density of bubble surface

- spherical symmetry
  dynamics of a bubble wall → 1-dim dynamics

- no gravity
  To perform analytic computation





picture: T. Hayashi

Setup

2. Formalism

- action of a thin wall bubble (Polyakov action)

$$S_{\rm P}[X^\mu, \gamma^{ab}] = -\sigma \int_{\partial\mathcal{B}} d^3x \sqrt{-\gamma}\frac{1}{2}\left[\gamma^{ab}\partial_a X^\mu \partial_b X_\mu - 1\right] + \Delta V \int_{\mathcal{B}} d^4X\sqrt{-g}$$

bubble wall — bulk (bubble interior)

- spherical bubble

$$\{X^\mu(x^a)\} \to \{T(\tau), R(\tau), \theta, \varphi\} \qquad \gamma_{ab}dx^a dx^b = -N^2 d\tau^2 + R^2(\tau)d\Omega^2$$

τ :proper time on the wall

gauge fixing : $dN/d\tau = 0$

$$S_{\rm P}[T, R, N] = 4\pi\sigma \int \frac{d\tau}{2\pi}\left\{\frac{1}{2}R^2\left[N^{-1}\left(-f\dot{T}^2 + f^{-1}\dot{R}^2\right) - N\right] + \rho_0^{-1}R^3\dot{T}\right\}$$

$$\rho_0 \equiv \frac{3\sigma}{\Delta V}$$

$$G(R=0, R=R_1) = \int_0^\infty dN \int_{R(\tau=0)=0}^{R(\tau=1)=R_1} \mathcal{D}T\mathcal{D}R \exp(iS_{\mathrm{P}}[T,R,N])$$

take into account every proper duration

quantum-mechanical amplitude from a zero-size bubble to finite one

$$\int_0^\infty dN \, A(N) \exp\left(iS_{\mathrm{P}}[\bar{T},\bar{R},N]\right)$$

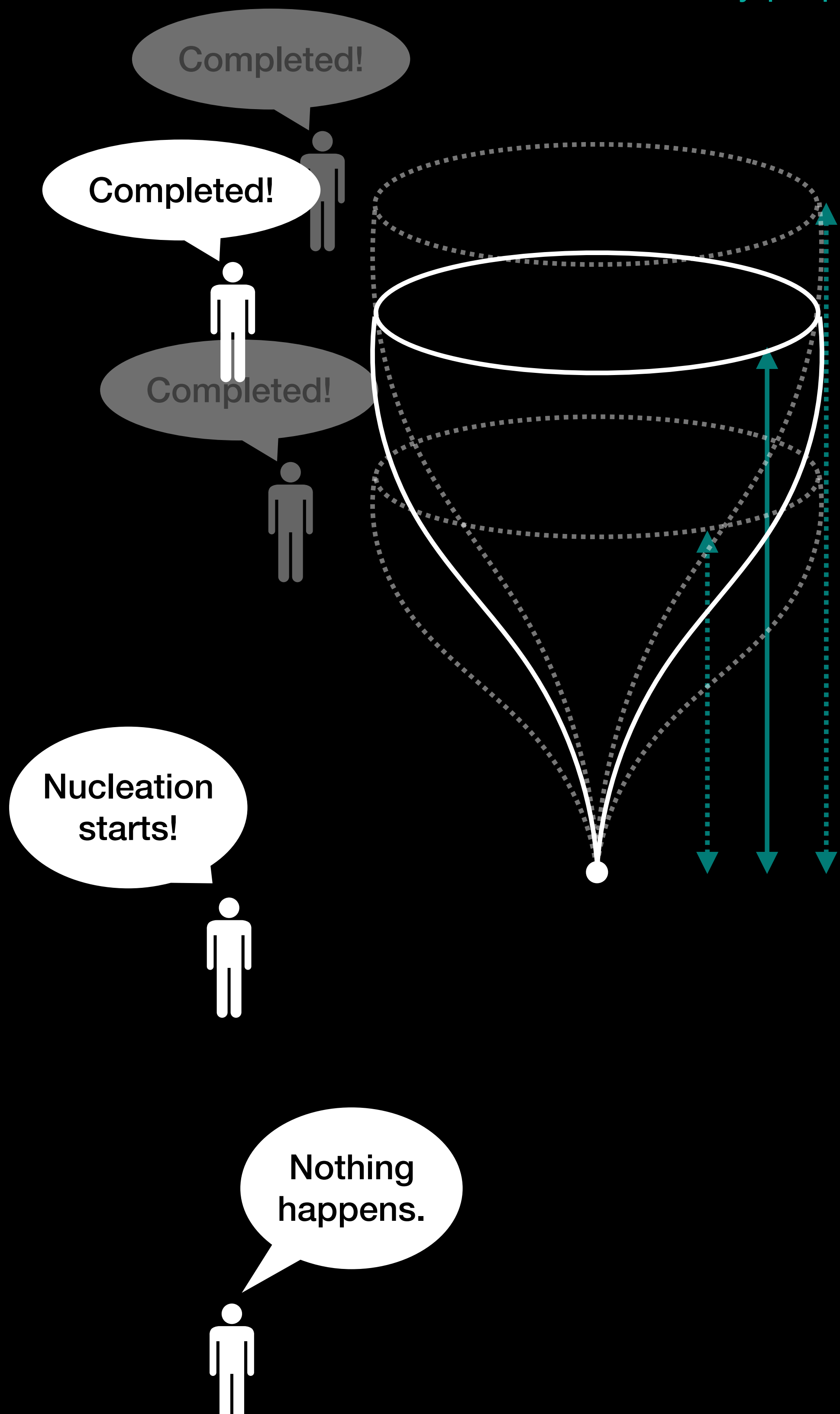
quantum corrections
+zero modes

leading contribution
(exponential suppression)

oscillatory integrand

$T = \bar{T}$ and $R = \bar{R}$

Solution of the E.O.M.



lapse N controls
the proper duration

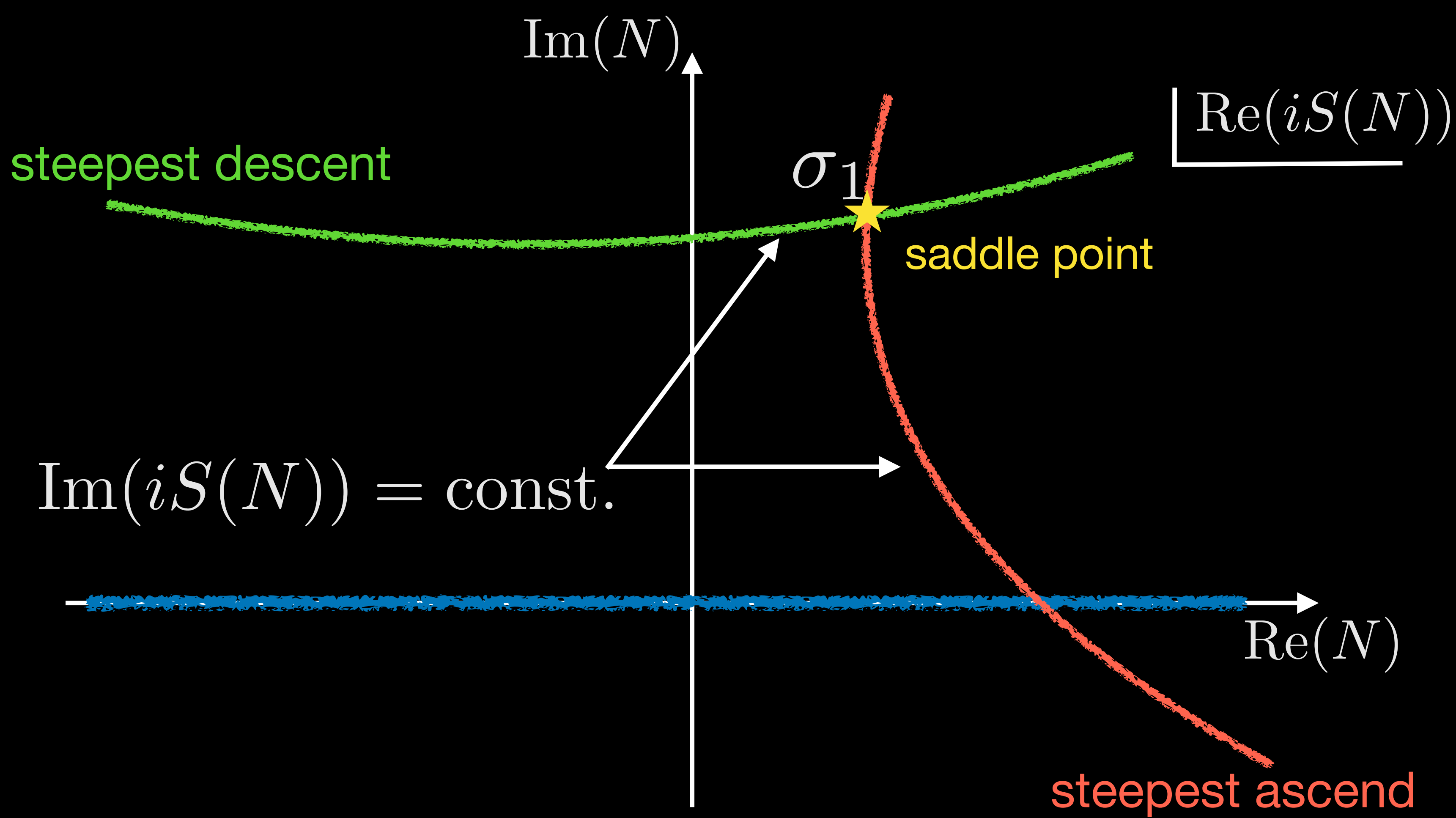
$$\Delta t = N \Delta\tau$$

$$\Delta\tau = 1$$

Picard-Lefschetz theory

$$\int_{\mathbb{R}} dN \exp\left(iS(N)\right) \longrightarrow \int_{\mathcal{C}} dN \exp\left(iS(N)\right)$$

absolutely convergent!!
(No oscillatory integrand!!)



I will show this with an easy example -> my PDF notes

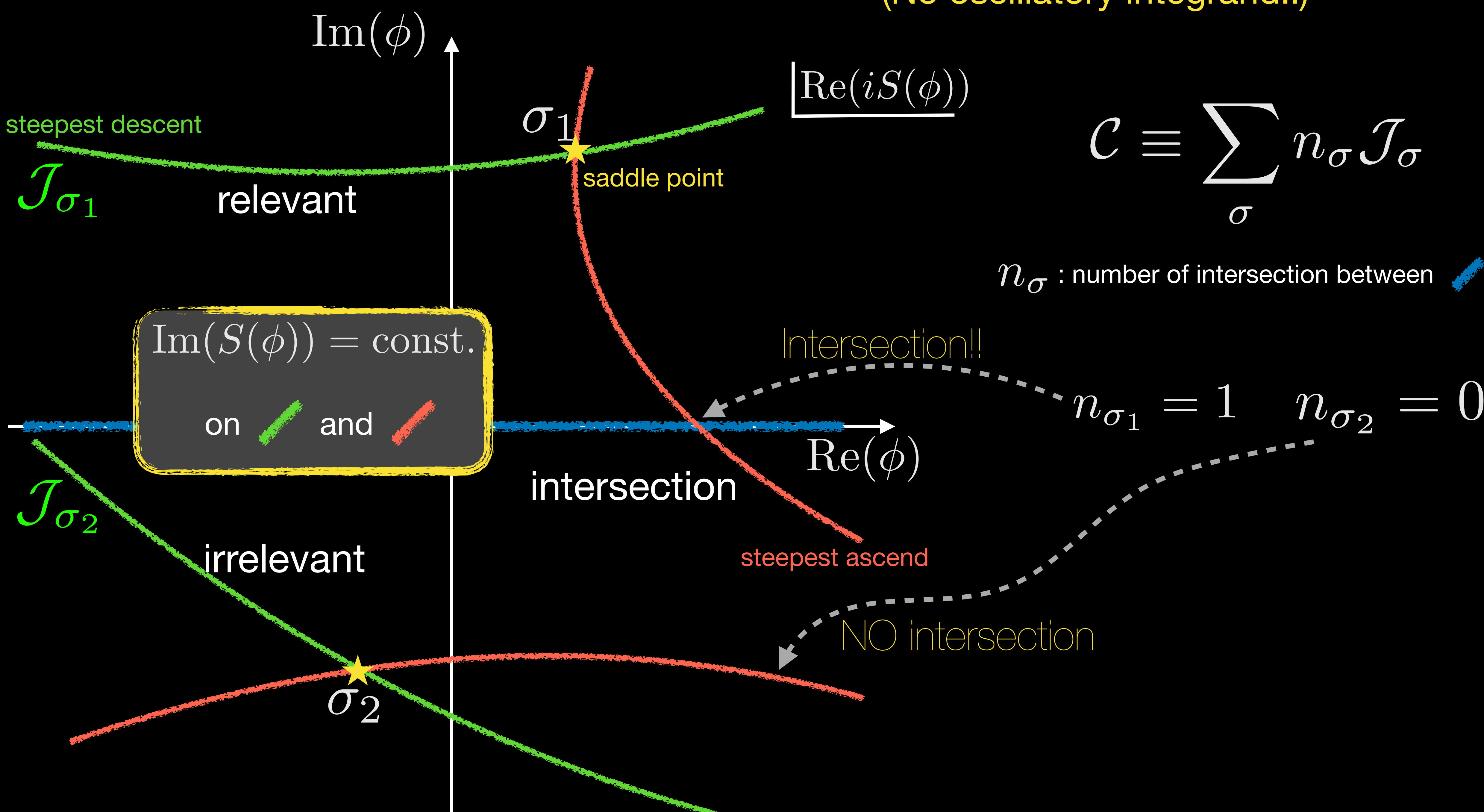
Picard-Lefschetz theory

$$\int_{\mathbb{R}} d\phi \exp\left(iS(\phi)\right) \longrightarrow \int_{\mathcal{C}} d\phi \exp\left(iS(\phi)\right)$$

absolutely convergent!!
(No oscillatory integrand!!)

$$\mathcal{C} \equiv \sum_{\sigma} n_\sigma \mathcal{J}_\sigma$$

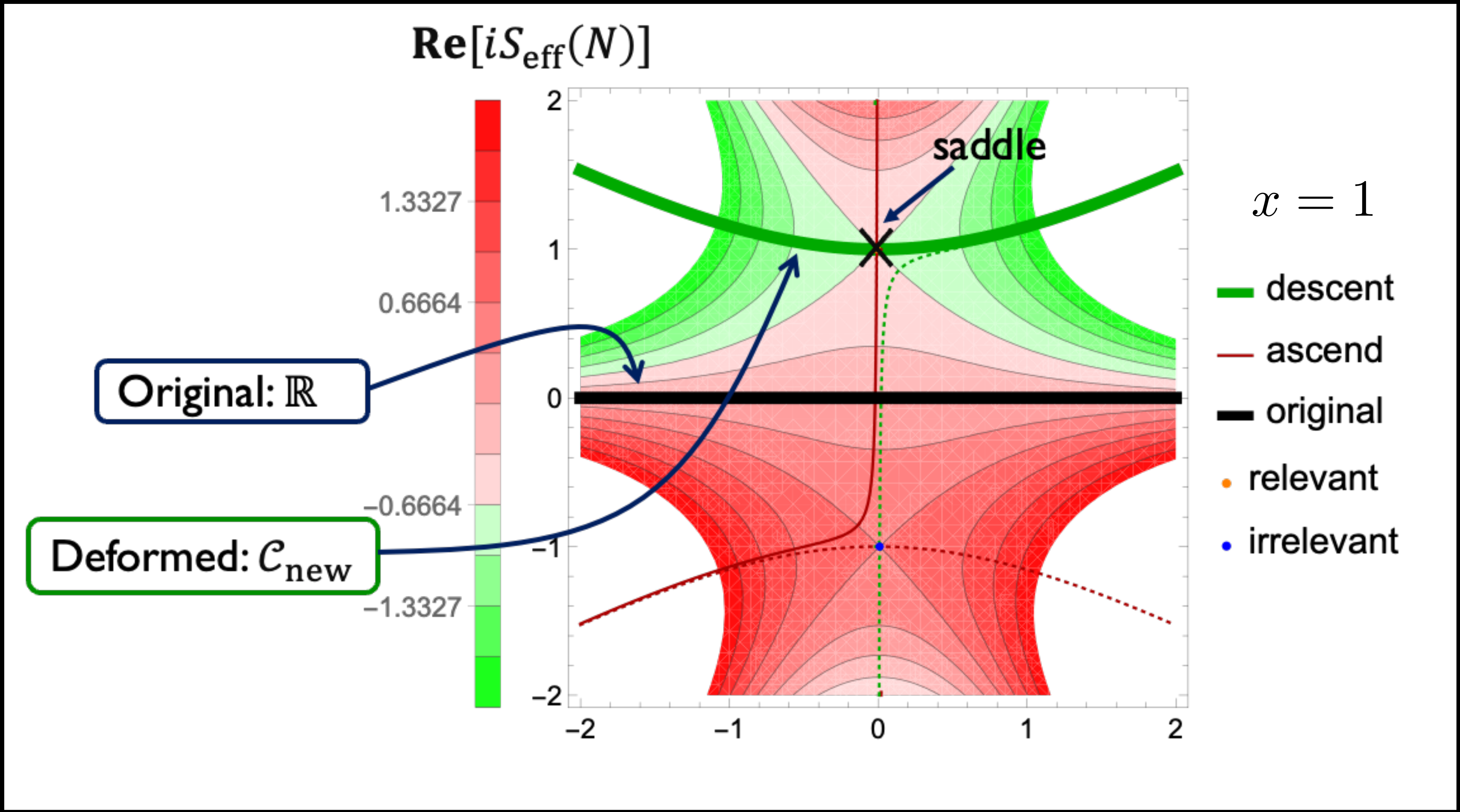
n_σ : number of intersection between and



Picard-Lefschetz theory -example-

$$\mathrm{Ai}(x) = \int_{\mathbb{R}} dN \exp[iS_{\text{eff}}(N,x)] \qquad S_{\text{eff}}(N,x) = \frac{N^3}{3} + xN$$

$$\simeq \frac{\sqrt{\pi}}{x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right) \text{ for } |\arg(x)| < \frac{2\pi}{3} \text{ and } |x| \to \infty$$



$$\int_0^\infty dN\, A(N) \exp\left(iS_{\mathrm{P}}[\bar{T}, \bar{R}, N]\right)$$

oscillatory integrand

$$\downarrow$$

$$\int_{\mathcal{C}} dN\, A(N) \exp\left(iS_{\mathrm{P}}[\bar{T}, \bar{R}, N]\right)$$

Lefschetz thimbles

~ Gaussian!!

$$\exp\left(iS_{\rm P}[\bar{T}, \bar{R}, N]\right)$$

Integrated E.O.M.

$$-2\pi\sigma R^2\left(N^{-2}(f\dot{T}^2 - f^{-1}\dot{R}^2) - 1\right) = H \qquad 4\pi\sigma R^2 f(R)\left(\frac{\dot{T}}{N} - \frac{R}{\rho_0 f(R)}\right) = E$$

Integration constant (H) — fix H so that B.C. is satisfied

bubble's total energy (E) — set to zero

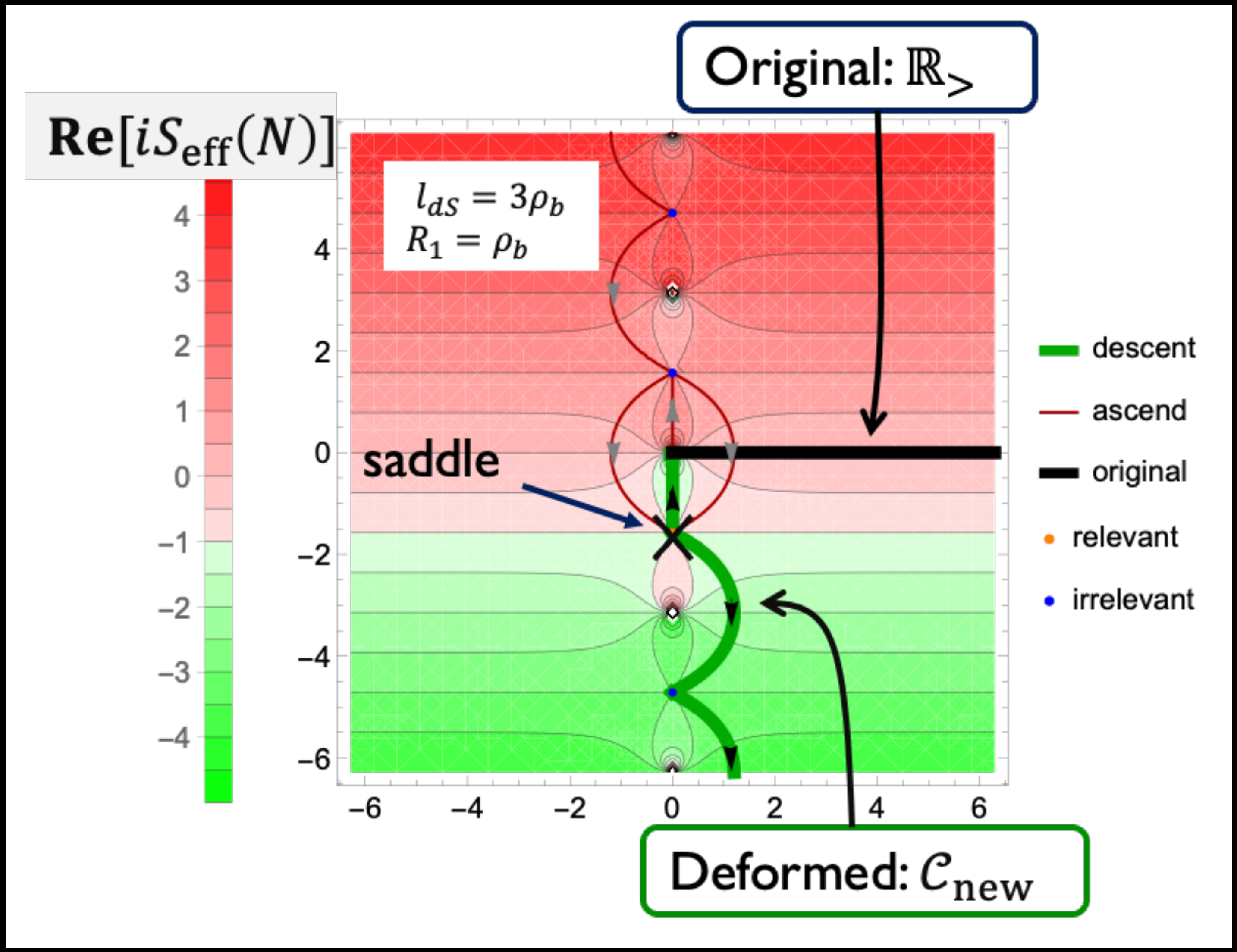
boundary condition on $R = \bar{R}$ (and $T = \bar{T}$)

Coleman's bubble	$R(\tau = 0) = 0$ and $R(1) = \rho_b$
large bubble	$R(0) = 0$ and $R(1) > \rho_b$
small bubble	$R(0) = 0$ and $R(1) < \rho_b$

boundary condition changes the structure of $S_{\rm P}(\bar{T}, \bar{R}, N)$

e.g. position of the saddle points in N-space

Coleman's bubble



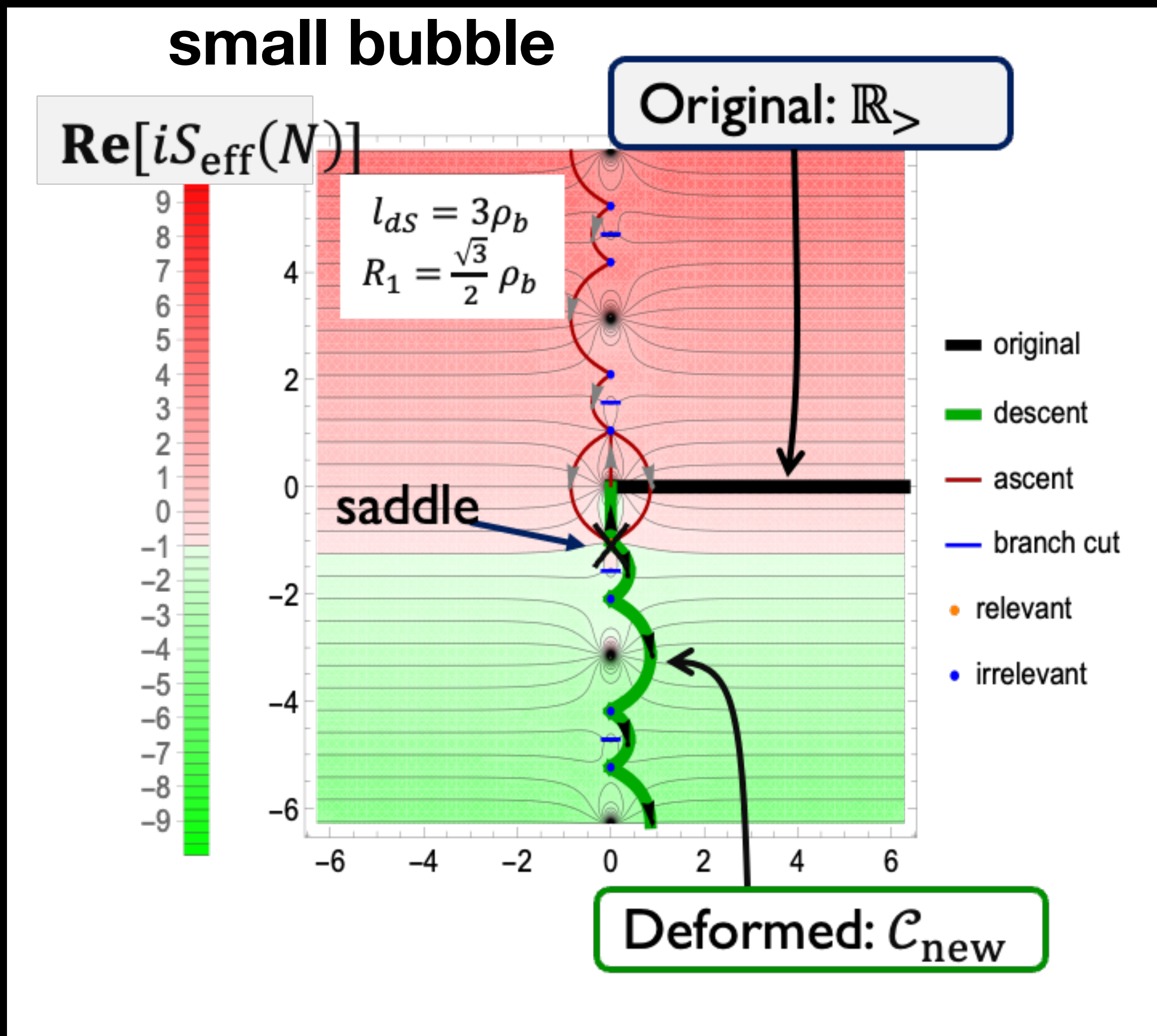
$$S_{\text{P}}[\bar{T}, \bar{R}, N] = \frac{2\pi\sigma\rho_b^3}{(1+\rho_b/\rho_0)^2}\left[\coth\frac{N}{\rho_b} - \frac{N}{\rho_b}\right]$$

$$G(R=0, R=\rho_b) \sim \int_0^\infty dN \exp\left(iS_{\text{P}}(N)\right) = \int_{\mathcal{C}_{\text{new}}} dN \exp\left(iS_{\text{P}}(N)\right) \sim \exp\left(-\frac{\pi^2\sigma\rho_b^3}{(1+\rho_b/\rho_0)^2}\right)$$

$$\frac{B_{\text{Coleman}}}{2}$$

$$P_{\text{Decay}} = |G(R=0, R=\rho_b)|^2 \sim \exp\left(-B_{\text{Coleman}}\right)$$

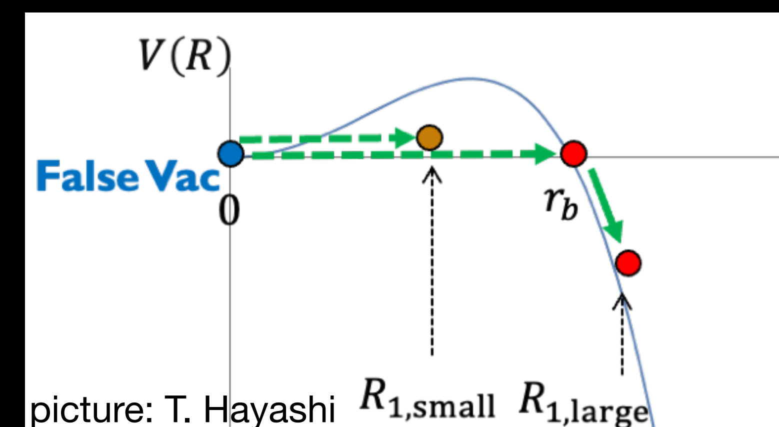
consistent with the standard result!

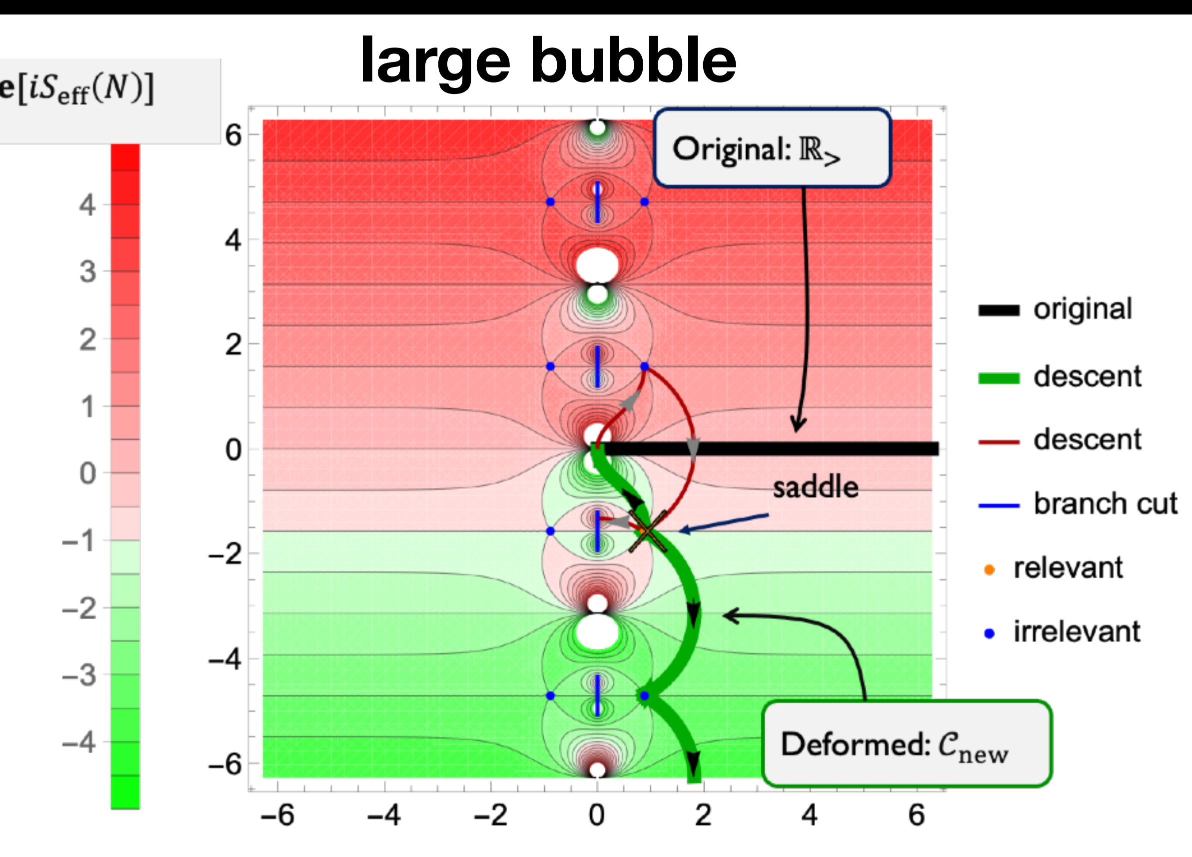


$$G(R=0, R=R_{\mathrm{bubble}})$$
$$R_{\mathrm{bubble}} < \rho_b$$
$$\sim \int_0^\infty dN \exp\left(iS_{\mathrm{P}}(N)\right) \sim \exp\left(-B_{\mathrm{small}}/2\right)$$

$$B_{\mathrm{small}} < B_{\mathrm{Coleman}}$$

A small bubble may have a chance to expand due to the external source e.g. thermal radiation inside the bubble, external gravitational force etc..





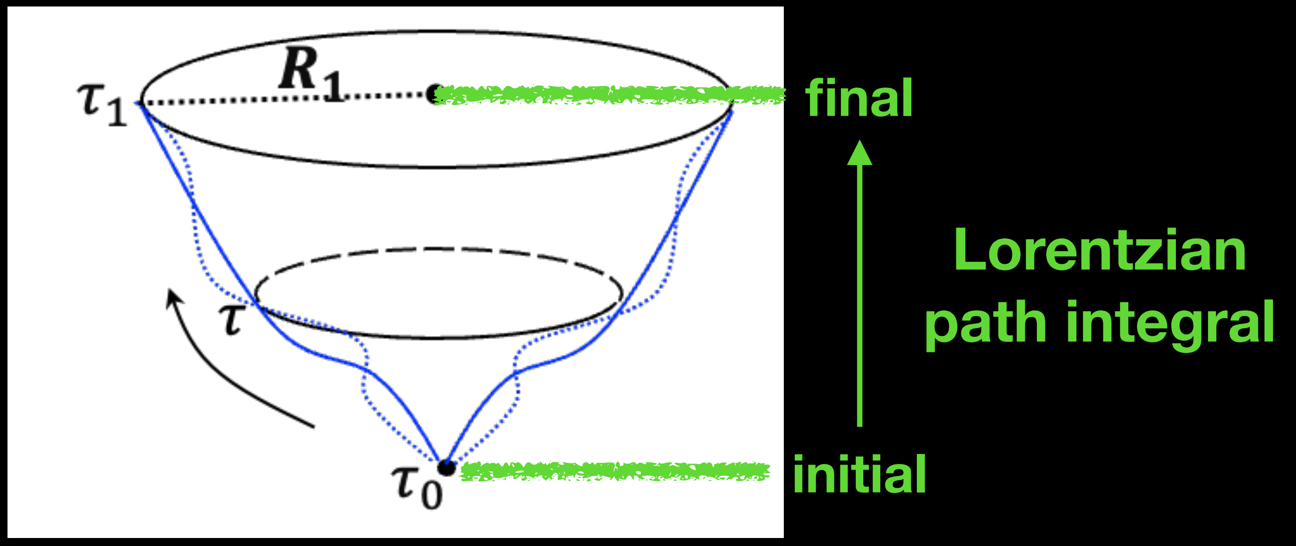
$$G(R=0, R=R_{\mathrm{bubble}})$$
$$R_{\mathrm{bubble}} > \rho_b$$
$$\sim \int_0^\infty dN \exp\left(iS_{\mathrm{P}}(N)\right) \sim \exp\left(-B_{\mathrm{large}}/2 + i\theta(\rho_b, R_{\mathrm{bubble}})\right)$$

$$B_{\mathrm{large}} = B_{\mathrm{Coleman}}$$

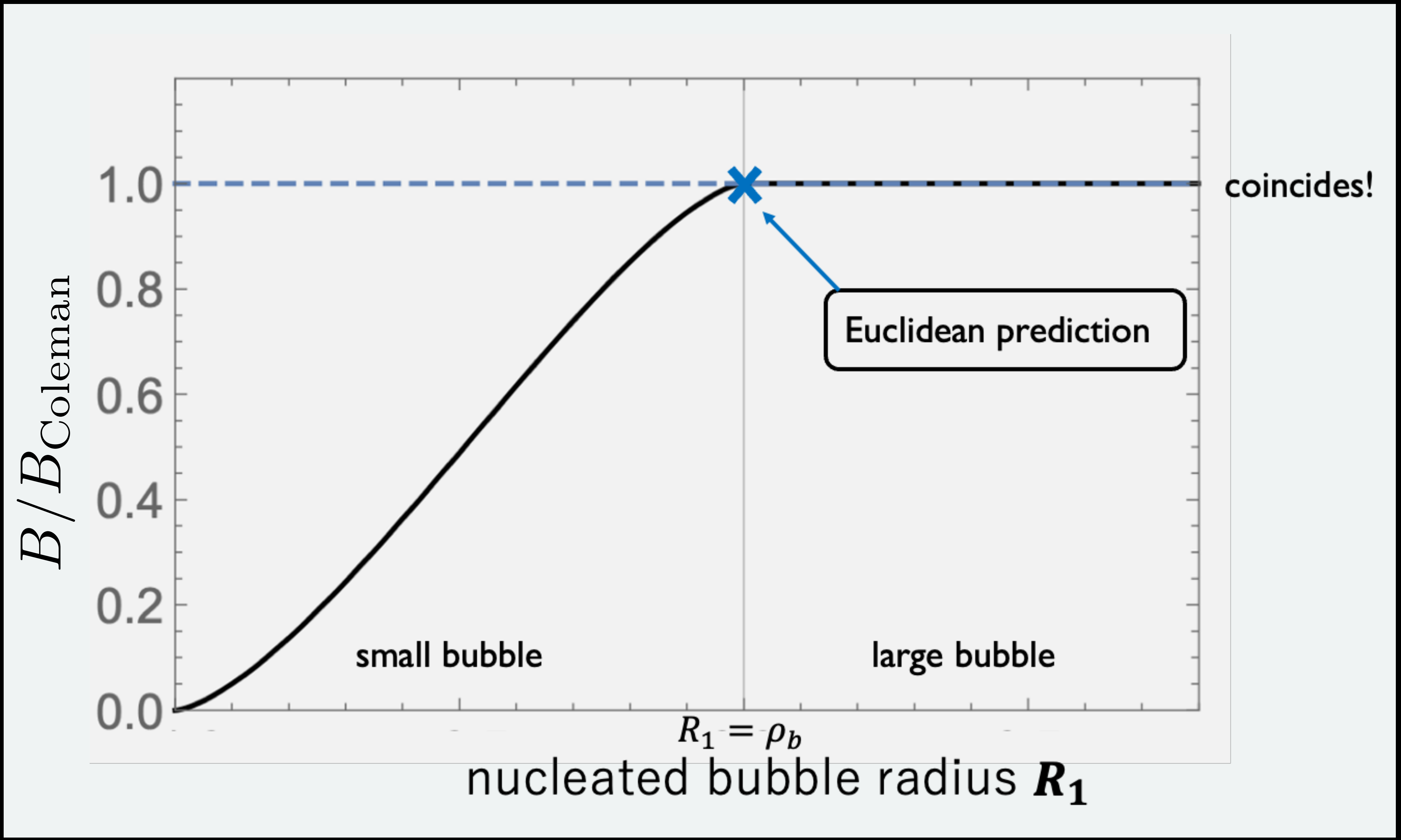
phase rotation caused by the classical bubble expansion

$$R = \rho_b \to R = R_{\mathrm{bubble}}$$

$$\Gamma \sim A e^{-B}$$



picture: T. Hayashi



Summary

We formulated the nucleation of a vacuum bubble in the Lorentzian path integral.

We demonstrated the nucleation of a bubble of arbitrary size and found:

- exponent of the nucleation rate of a bubble of *standard size* agrees with the Coleman's result.
- large bubble nucleation can be interpreted as **nucleation of standard-size bubble + classical expansion**
- small bubble nucleates with higher probability

Our strategy is useful to search for other vacuum decay processes that are more probable than the known process.
An arbitrary vacuum bubble can be set as the final state unlike the bounce calculation.

Still, some restrictions are imposed: no gravity and thin wall approximation.