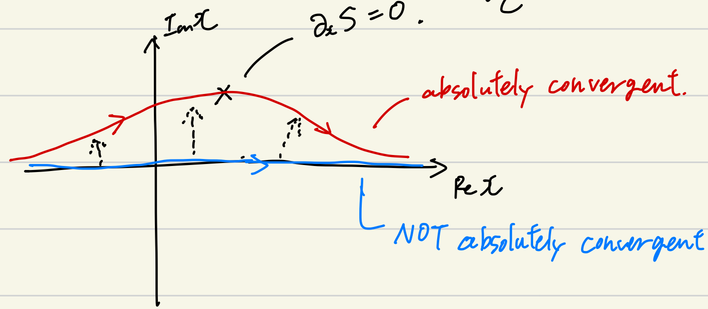
$$\int_{\mathbb{R}} e^{iS[x]}\, dx \Rightarrow \int_C e^{iS[x]}\, dx.$$



steepest descent contours.
leading to convergent integral. = Lefschetz thimbles $\mathcal{J}_\sigma$.

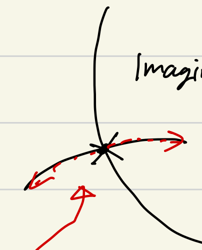
$$\int_R e^I \, Dx$$

$$S[x] = x^2, \quad I \equiv iS/\hbar.$$

$$I = Re(I) + i\,Im(I) = h + iH.$$

$$x = u^1 + i u^2.$$

Downward flow : $$\frac{du^i}{d\lambda} = -g^{ij}\frac{\partial h}{\partial u^i}$$



λ : a parameter along the flow.

g_{ij}: a Riemann metric introduced on the complex plane.

$$ds^2 = |dx|^2 = du\, d\bar{u} \qquad (u = u^1 + iu^2,\ \bar{u} = u^1 - iu^2).$$

$$g_{\bar{u}\bar{u}} = g_{uu} = 0 \qquad g_{u\bar{u}} = g_{\bar{u}u} = 1/2. \qquad h = (I + \bar{I})/2.$$

※: $I = I[x] = I[u]$
$\bar{I} = \bar{I}[\bar{x}] = I[\bar{u}]$.

$$\frac{du^i}{d\lambda} = -g^{ij}\frac{\partial h}{\partial u^j} \searrow \begin{cases} \dfrac{du}{d\lambda} = -2\dfrac{\partial h}{\partial \bar{u}} = -\dfrac{\partial \bar{I}}{\partial \bar{u}} \\[2ex] \dfrac{d\bar{u}}{d\lambda} = -2\dfrac{\partial h}{\partial u} = -\dfrac{\partial I}{\partial u} \end{cases}$$

Then, imaginary part H $(= (I - \bar{I})/(2i))$.

$$\frac{dH}{d\lambda} = \frac{1}{2i}\frac{d(I - \bar{I})}{d\lambda} = \frac{1}{2i}\left(\frac{\partial I}{\partial u}\frac{du}{d\lambda} - \frac{\partial \bar{I}}{\partial \bar{u}}\frac{d\bar{u}}{d\lambda}\right) = 0.\ !$$