Functional determinants for radial operators

$$\mathcal{M} = -\Delta + V(r)$$
$$\mathcal{M}^{free} = -\Delta$$

<u>$d=1$:</u>

$$\frac{\det[\mathcal{M}+m^2]}{\det[\mathcal{M}^{free}+m^2]} = \frac{\psi(\infty)}{\psi^{free}(\infty)}$$

$$[\mathcal{M}+m^2]\psi = 0$$
$$\psi(0)=0 \qquad \psi'(0)=1$$

<u>$d>1$:</u>

$$\Phi(r,\theta) = \frac{1}{r^{\frac{d-1}{2}}}\,\underline{\psi_{(l)}(r)}\;\underline{Y_{(l)}(\bar{\theta})}$$

$$\mathcal{M}_{(l)} = -\frac{d^2}{dr^2} + \frac{\left(l+\frac{d-3}{2}\right)\left(l+\frac{d-1}{2}\right)}{r^2} + V(r)$$

$$d = \frac{(2l+d-2)(l+d-3)!}{l!\,(d-2)!}$$

$$\ln\left(\frac{\det[\mathcal{M}+m^2]}{\det[\mathcal{M}_{free}+m^2]}\right) = \sum_{l=0}^{\infty} \deg(l;d)\, \ln\left(\frac{\det[\mathcal{M}_{(l)}+m^2]}{\det[\mathcal{M}^{free}_{(l)}+m^2]}\right)$$

* ζ-function

$$\zeta_{\mathcal{M}}(s) = \mathrm{Tr}\left\{\frac{1}{\mathcal{M}^s}\right\} = \sum_{\lambda} \frac{1}{\lambda^s}$$

$$\zeta_{[\mathcal{M}+m^2]/\mu^2}(s) = \mu^{2s}\, \zeta_{[\mathcal{M}+m^2]}(s)$$

$$= \mu^{2s} \sum_{\lambda} \frac{1}{(\lambda+m^2)^s}$$

$$\ln\left(\det\left[\frac{\mathcal{M}+m^2}{\mu^2}\right]\right) \equiv -\zeta'_{[\mathcal{M}+m^2]/\mu^2}(0)$$

$$\shortparallel$$

$$\zeta'_{\mathcal{M}}(s) = -\sum_{\lambda} \frac{\ln\lambda}{\lambda^s} \qquad \boxed{\begin{aligned} &-\ln(\mu^2)\,\zeta_{[\mathcal{M}+m^2]}(0) \\ &\qquad -\zeta'_{[\mathcal{M}+m^2]}(0) \end{aligned}}$$

$$\zeta'_{\mathcal{M}}(0) = -\sum_{\lambda} \ln\lambda$$

$$\zeta(s) \equiv \zeta_{[\mathcal{M}+m^2]}(s) - \zeta_{[\mathcal{M}_{free}+m^2]}(s)$$

$\rightsquigarrow$ need to compute : $\zeta'(0)$, $\zeta(0)$

@ $s=0$ $\qquad \zeta_{\mathcal{P}}(0) = a_{d/2}(\mathcal{P})$

$$\mathcal{P} = -\Delta - E$$

$$E = -V(r) - m^2$$

$$\zeta(0) = \begin{cases} -\frac{1}{2}\int_0^\infty dr\, r\, V(r) & d=2 \\ 0 & d=3 \\ \frac{1}{16}\int_0^\infty dr\, r^3\, V(V+2m^2) & d=4 \end{cases}$$

$\underline{\zeta'(0):}$ Jost functions

$$M_{(\ell)}\, \underline{\phi_{(\ell),p}} = p^2 \phi_{(\ell),p}$$

$r \to 0 \qquad \phi_{(\ell),p} \sim \hat{j}_{\ell+\frac{d-3}{2}}(pr)$

$r \to \infty : \qquad \phi_{(\ell),p} \sim \frac{i}{2}\Big[\underline{J_\ell(p)}\, \hat{h}^{-}_{\ell+\frac{d-3}{2}}(pr)$

$$- J_\ell^*(p)\, \hat{h}^{+}_{\ell+\frac{d-3}{2}}(pr)\Big]$$

$$V(r) \overset{r\to\infty}{\sim} \begin{cases} r^{-2-\varepsilon} & d=2,3 \\ r^{-4-\varepsilon} & d=4 \end{cases}$$

$$\hat{h}^{\pm}_{l+\frac{d-3}{2}} \overset{r\to\infty}{\sim} e^{\pm ipr} \leftarrow$$

$$p^2 = -m^2 \Rightarrow p = im$$

$$\psi_{(l),im}(r) \sim J_l(im)\, e^{mr} \qquad r\to\infty$$

$$\psi_{(l)im}(r) \sim J_l^{free}(im)\, e^{mr}$$

$$\frac{\psi_{(l),im}(r)}{\psi^{free}_{(l),im}(r)} = \frac{J_l(im)}{J_l^{free}(im)} \equiv f_{(l)}(im)$$

normalized Jost func.

$$\parallel$$

$$\frac{\psi_{(l)}(\infty)}{\psi^{free}_{(l)}(\infty)} = \frac{\det(M_{(l)}+m^2)}{\det(M^{free}_{(l)}+m^2)}$$

by contour manipulations:

$$\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=0}^{\infty} \deg(l;d) \int_m^{\infty} dk\, [k^2-m^2]^{-s} \frac{\partial}{\partial k} \ln f_l(ik)$$

(*)

valid $\mathrm{Re}(s) > \frac{d}{2}$

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^{\infty} d\lambda\, \lambda^{-s} \frac{d \ln F(\lambda)}{d\lambda}$$

if (*) analytic in $s=0$:

$$\Rightarrow \quad \zeta'(0) = -\sum_{l=0}^{\infty} \deg(l;d) \ln f_l(im)$$

$$\zeta(s) = \zeta_f(s) + \zeta_{as}(s)$$

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \sum_{l=0}^{\infty} \deg(l;d) \int_m^{\infty} du\, [u^2 - m^2]^{-s} \frac{\partial}{\partial u} \left[\ln f_l(iu) - \ln f_l^{asym}(iu) \right]$$

$$\zeta_{as}(s) = \frac{\sin \pi s}{\pi} \sum_{l=0}^{\infty} \deg(l;d) \int_m^{\infty} du\, [u^2 - m^2]^{-s} \frac{\partial}{\partial u} \ln f_l^{asym}(iu)$$

$\nu \to \infty$, $u \to \infty$, $\frac{u}{\nu}$ fixed

$$f(x,\nu) \overset{x\to 0}{\sim} \sum_{n=0}^{N} a_n(\nu)\, x^n + \mathcal{O}(x^{N+1})$$

indep. ν

$\forall \varepsilon > 0 \quad \exists N$

$$\left| f(x,\nu) - \sum_{n=0}^{N} a_n(\nu)\, x^n \right| < \varepsilon \quad \forall \nu$$

*) Asymptotics of Jost functions

$$\mathcal{M}_{(\ell)}\, \phi_{(\ell),p} = p^2 \phi_{(\ell),p}$$

$$\phi^{\text{free}}_{(\ell),p} = \hat{J}_{\ell+\frac{d-3}{2}}(pr)$$

integral form:

$$\phi_{(\ell),p}(r) = \phi^{\text{free}}_{(\ell),p} + \int_0^r dr'\, G_{\ell p}(r,r')\, V(r')\, \phi_{\ell,p}(r')$$

$$\Rightarrow f_\ell(ik) = 1 + \int_0^\infty dr\; r\, V(r)\, \phi_{(\ell),ik}(r)\, K_{\ell+\frac{d}{2}-1}(kr)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \dots$$

$$\nu = l + \frac{d}{2} - 1$$

$$\ln f_l(ik) = \int_0^\infty dr\, r\, V(r)\, K_\nu(kr)\, I_\nu(kr)$$

$$- \int_0^\infty dr\, r\, V(r)\, K_\nu^2(kr) \int_0^r dr'\, r'\, V(r')\, I_\nu^2(kr') + \mathcal{O}(V^3)$$

$$I_\nu(kr) K_\nu(kr) \sim \frac{\alpha}{\nu} + \frac{\beta}{\nu^3} + \mathcal{O}\left(\frac{1}{\nu^4}\right)$$

$$I_\nu(kr') K_\nu(kr) \sim \frac{\gamma}{\nu} e^{-\nu\delta} \left[1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right]$$

$\nu \to \infty$, $k \to \infty$ $\frac{k}{\nu}$ fixed

we define $\ln f_l^{asym}(ik)$ $\mathcal{O}(V)$ & $\mathcal{O}(V^2)$

•) $\zeta_P'(0)$,

$$\Rightarrow \zeta_P'(0) = - \sum_{l=0}^\infty \deg(l;d) \left(\ln f_l(ik) - \ln f_l^{asym}(ik)\right)$$

• $\zeta'_{as}(0)$:

$$d=4\,, \quad \nu = l+1 \qquad \deg(l;4) = \nu^2$$

$$R_1(s) \equiv \frac{\Gamma(s+\frac{1}{2})\, r^{2s}}{\Gamma(s)\Gamma(\frac{1}{2})} \underbrace{\sum_{\nu=1}^{\infty} \frac{\nu^{1-2s}}{\left(1+\left(\frac{\mu r}{\nu}\right)^2\right)^{s+\frac{1}{2}}}}$$

$$= \sum_{\nu=1}^{\infty} \nu^{1-2s} \left\{ \left(1+\left(\frac{\mu r}{\nu}\right)^2\right)^{-s-\frac{1}{2}} - 1 + \left(s+\tfrac{1}{2}\right)\left(\frac{\mu r}{\nu}\right)^2 \right\}$$

[illegible]

$$+ \underbrace{\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2s-1}}}_{\zeta_R(2s-1)} - \left(s+\tfrac{1}{2}\right)(\mu r)^2 \underbrace{\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2s+1}}}_{\zeta_R(2s+1)}$$

$$\uparrow\ s \to 0$$

$$\zeta_R(1)$$

$$\rightarrow \ln\left(\frac{r}{2}\right) + \gamma + 1$$

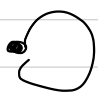
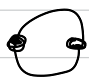
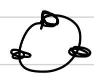
Comparison with Feynman diag. approach:

$$\ln\left(\frac{\det[\mathcal{M}+m^2]}{\det[\mathcal{M}^{\text{free}}+m^2]}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}A^{(k)}$$

$$\ln\left(\det\left(\frac{-\Delta+m^2+V}{-\Delta+m^2}\right)\right)$$

$$= \ln\det(1+\underbrace{GV}_{=T})$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\underbrace{\mathrm{Tr}(T^k)}_{A^k}$$

$=$  $-\frac{1}{2}$  $+\frac{1}{3}$  $+\dots$

$d=4$ $\quad A^{(1)}$ & $A^{(2)}$ divergent

$$A^{(1)} = \int d^dx\, V(x)\lim_{x\to y} G(x,y)$$

$$A^{(1)} = \sum_{\ell=0}^{\infty}\deg(\ell;d)\,[\ln f_\ell(\infty)]_{\mathcal{O}(V)}$$

$$A^{(2)} = \int d^d x \int d^d y \, V(x) \, G(x,y) \, V(y)$$

$$\Rightarrow A^{(2)} = -2 \sum_{\ell=0}^{\infty} \deg(\ell;d) \left[\ln f_\ell(i m)\right]_{\mathcal{O}(V^2)}$$

$$-\frac{1}{2} A^{(2)} \longleftrightarrow \mathcal{O}(V^2)$$

$$A^{(1)} \longleftrightarrow \mathcal{O}(V)$$

$$\ln\left(\frac{\det(M + m^2)}{\det[M^{\text{free}} + m^2]}\right) = \sum_{\ell=0}^{\infty} \deg(\ell;d) \Big\{ \ln f_\ell(im) - \left[\ln f_\ell(im)\right]_{\mathcal{O}(V)} - \left[\ln f_\ell(im)\right]_{\mathcal{O}(V^2)} \Big\}$$

$$+ A^{(1)}_{\text{fin}} - \frac{1}{2} A^{(2)}_{\text{fin}}$$

$$A^{(1)} = \int \frac{d^{4-\varepsilon}}{(2\pi)^{4-\varepsilon}} \frac{\tilde{V}(0)}{k^2 + m^2}$$

$$\tilde V(k) = \int d^4x \, V(x)\, e^{-ikx}$$

$$\Rightarrow A^{(1)} = -\frac{m^2}{16\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi + \underbrace{\ln\frac{\mu^2}{m^2}}_{\mu = m} + 1\right]\int d^4x \, V(x)$$

$$\overset{\overline{MS}}{\Longrightarrow} A^{(1)}_{fin} = -\frac{m^2}{8}\int_0^\infty dr \, r^3 \, V(r)$$

$$A^{(2)} = \int \frac{d^{4-\varepsilon}q}{(2\pi)^{4-\varepsilon}} \, |\tilde V(q)|^2 \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \, \frac{1}{(k^2+m^2)\left((k+q)^2+m^2\right)}$$

$$\Rightarrow A^{(2)} = \frac{1}{16\pi^2}\left[\frac{2}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2}\right. \int d^4x \, V(x)^2$$

$$+ \frac{1}{128\pi^4}\int dq \, q^3 \, |\tilde V(q)|^2 \left[2 - \frac{\sqrt{q^2+4m^2}}{q} \ln\left(\frac{\sqrt{q^2+m^2}+q}{\sqrt{q^2+m^2}-q}\right)\right.$$

$$\overset{\overline{MS}}{\Longrightarrow} A^{(2)}_{fin} = \frac{1}{128\pi^4}\int_0^\infty dq \, q^3 \, |\tilde V(q)|^2 \left[2 - \frac{\sqrt{q^2+4m^2}}{q} \ln\left(\frac{\sqrt{q^2+m^2}+q}{\sqrt{q^2+m^2}-q}\right)\right.$$

Feynman	δ-func
substract full	subtract only asympt
$O(V)$ & $O(V^2)$	