Dark Matter on the lattice

Yannick Dengler

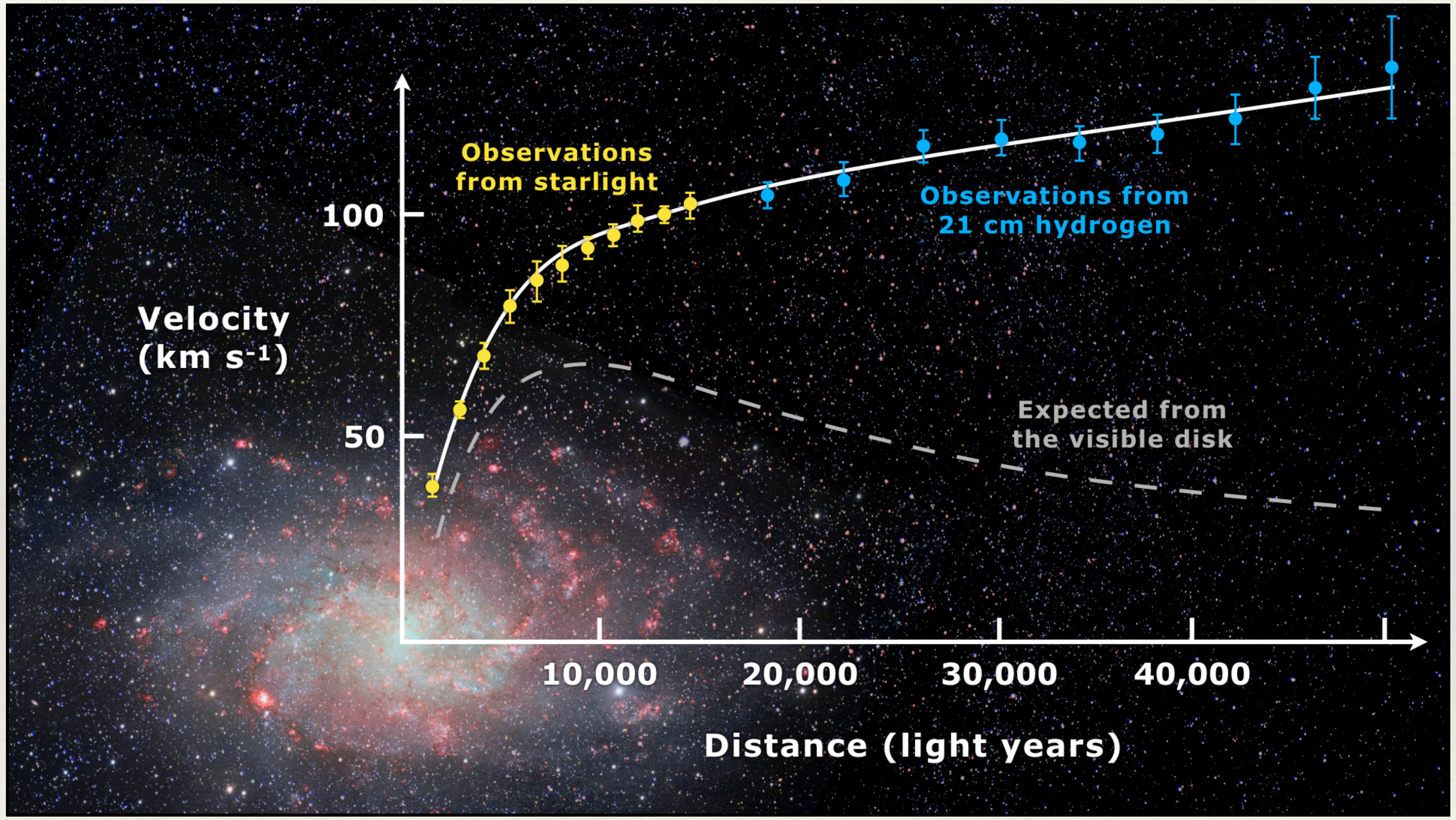
Outline

- Dark Matter
- Lattice Field theory
- Two examples

Observable	Application
Equation of state	Dark matter in neutron stars
Scattering: $2 \rightarrow 2$, $3 \rightarrow 2$	Small Scale Structure Relic density

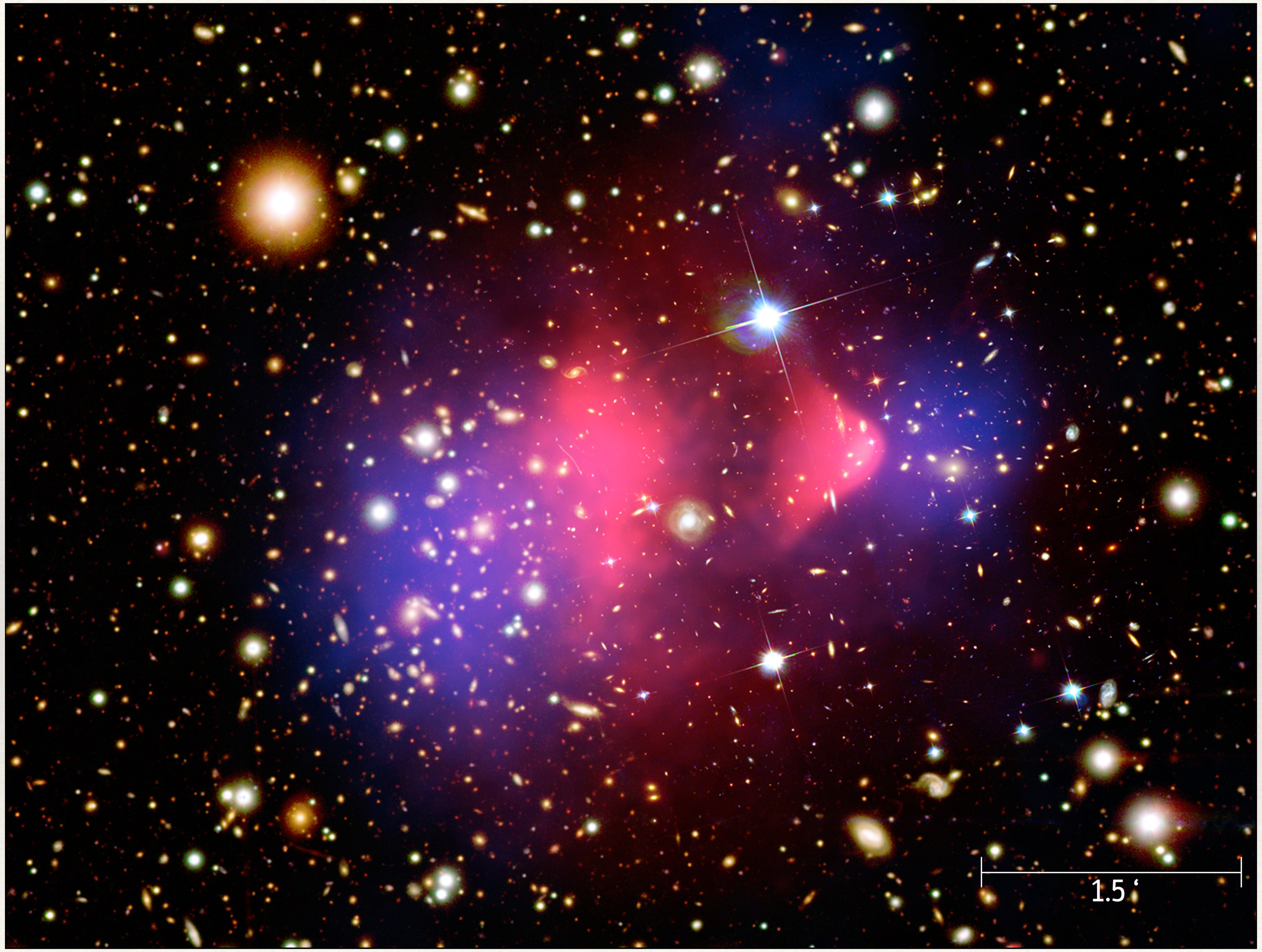
Dark Matter

- Collection of astronomical phenomena
  - Motion of objects, Large scale structure, gravitational lensing, …
  - No explanation in the standard model
- Explanations:
  - Modified Gravity
  - Particle *beyond the standard model*



Dark Matter Particle

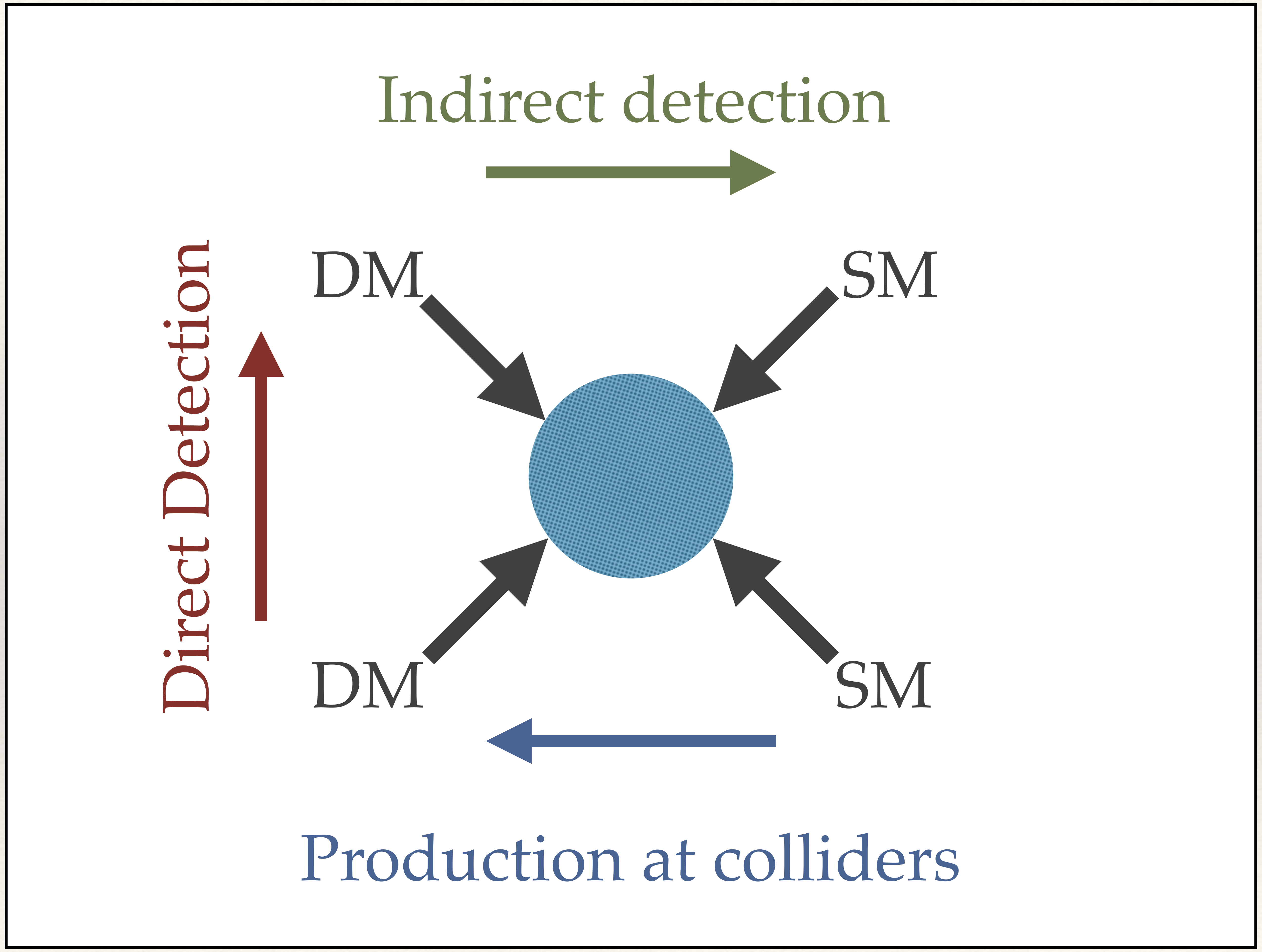
- Evidence for particle DM:
  - i.e. "Bullet cluster"
- Properties:
  - Massive, stable, "invisible"
- Interaction?
  - With SM: no (low)
  - Self: Maybe



Chandra X-ray Observatory

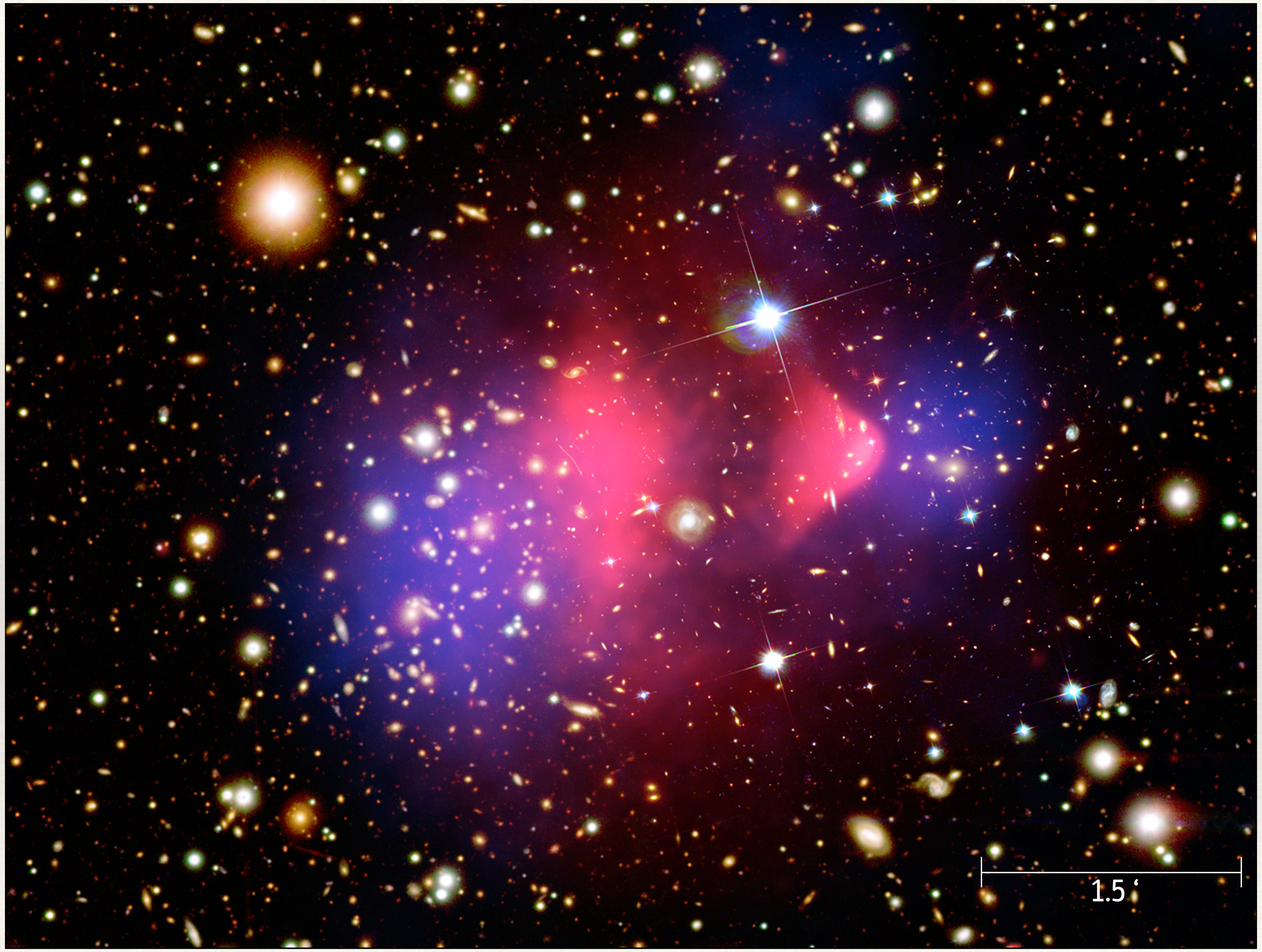
Dark matter searches

- The *standard* approach to dark matter searches
- Usually relies on some interaction with the standard model
  - DM without any SM interaction is still viable
- We can also learn about dark matter by only looking at a separate dark sector



What can lattice do?

- Test / limit effective theories
- Provide first-principles verification of dark matter models
- Use lattice data directly to make predictions or to compare to astro-data



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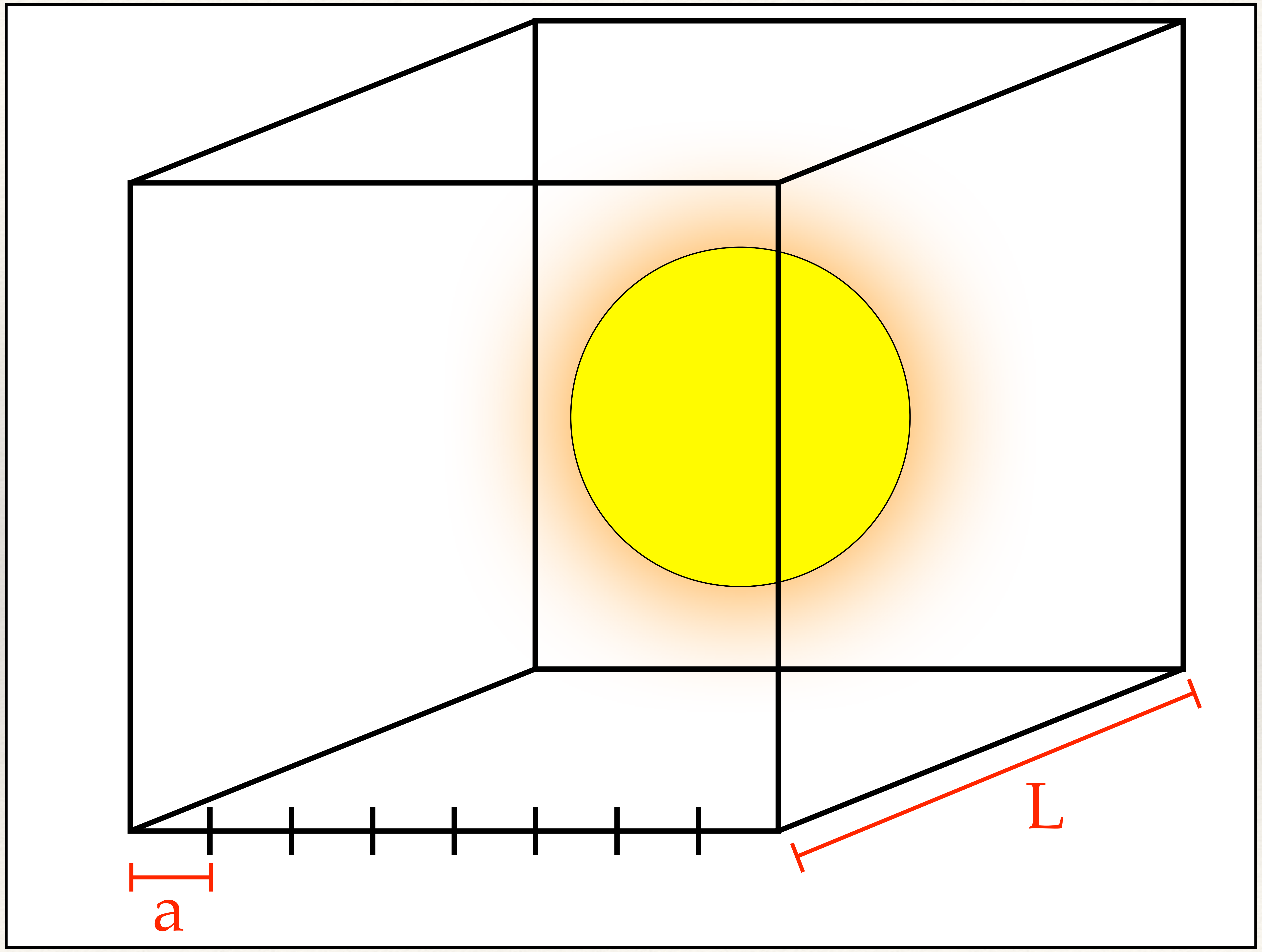
Lattice Field Theory

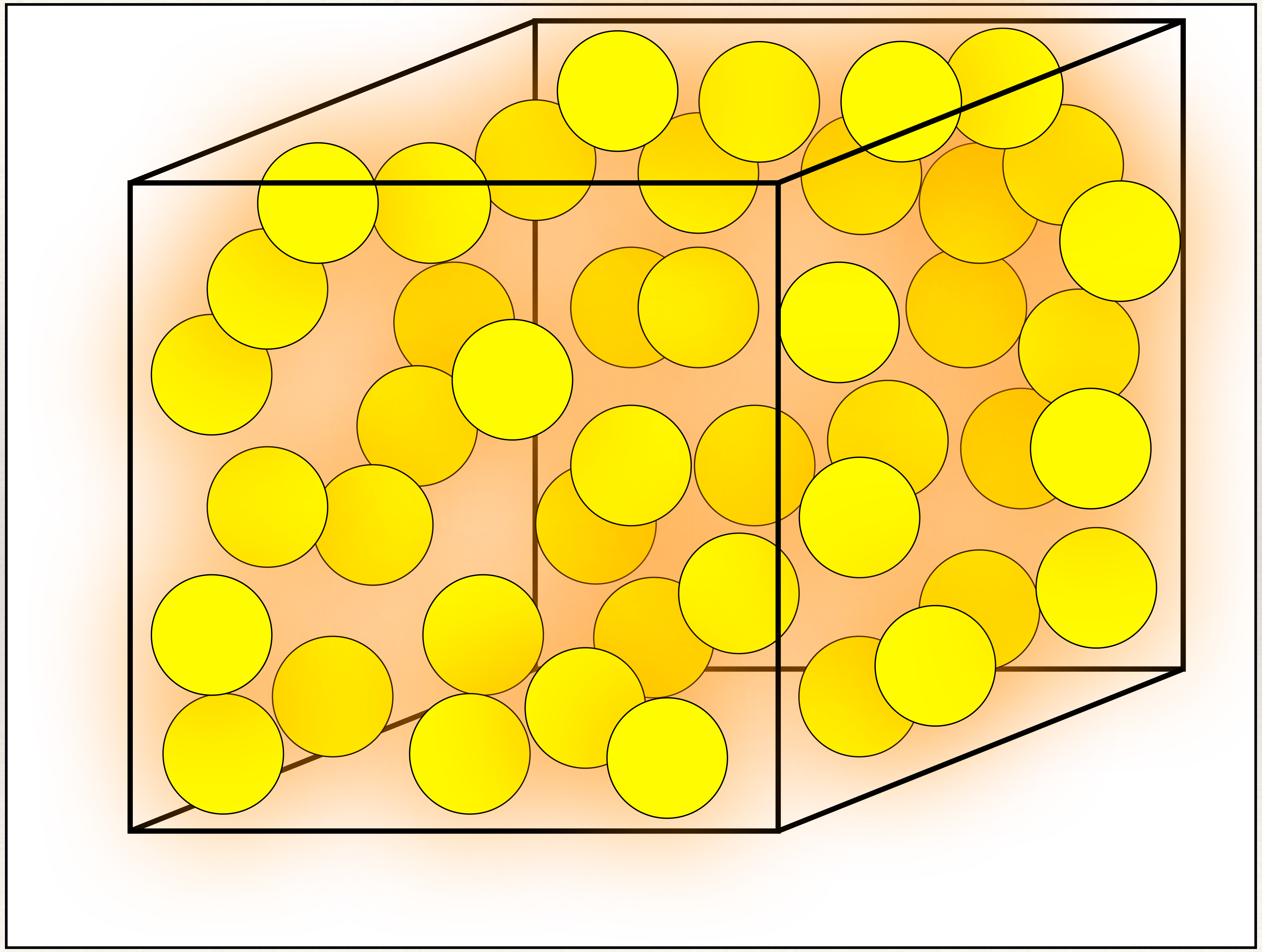
- Perform calculations on a discretized lattice with volume $a^4(N_L^3 N_T)$
  - Introduces IR and UV cutoff at L and a
  - Discretized rotational symmetry
- Importance sampling of gauge configurations via:
  - Probability interpretation of the action $p = \exp[-S(x)]$
- Imaginary time (Euclidean)



Gattringer, Lang

Sign Problem

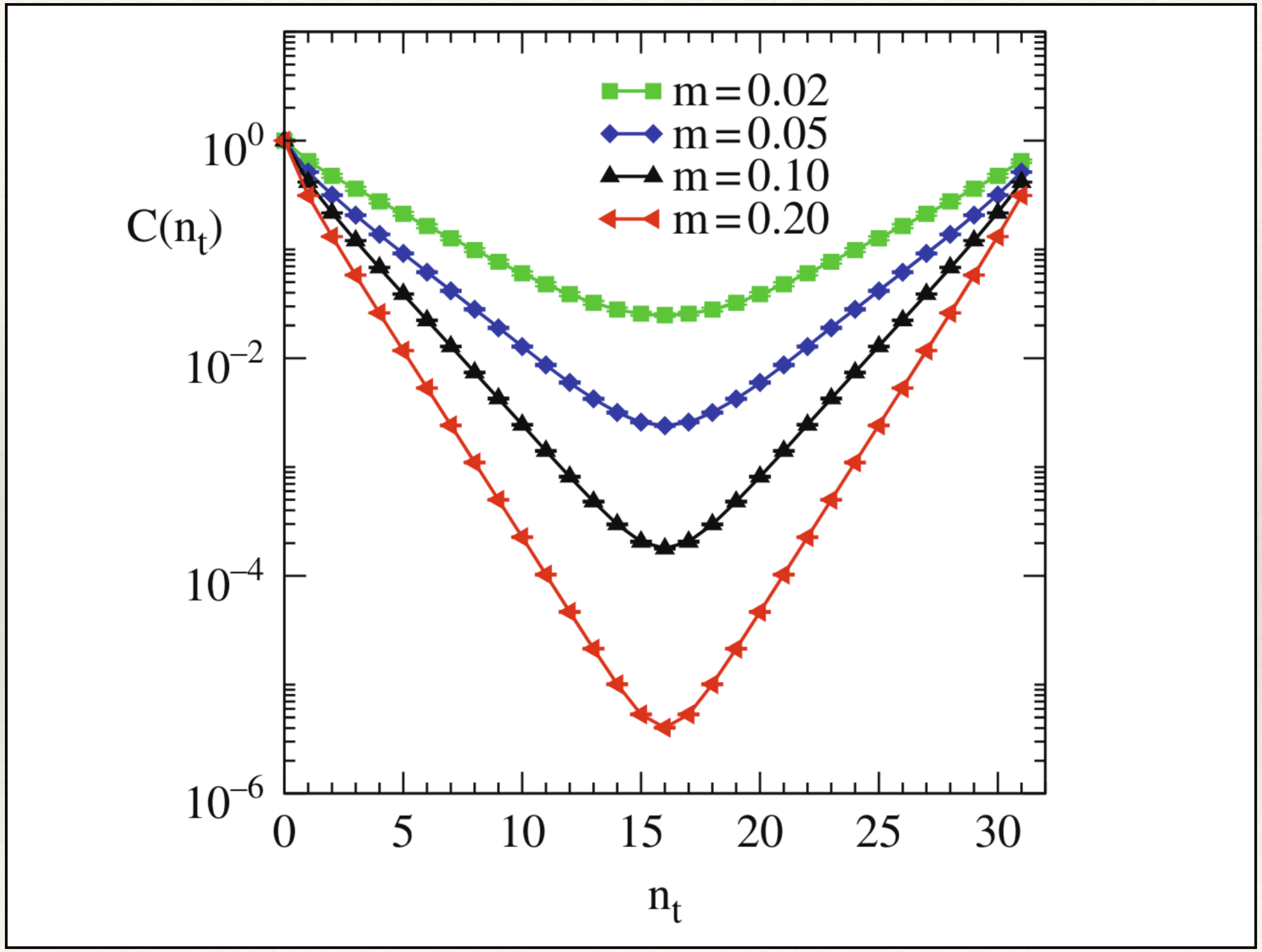
- Importance sampling: $p = \exp[-S(x)] \in \mathbb{R}$
- Add finite density to the action via chemical potential μ
  - Makes action complex for SU(3)
- Probability interpretation is lost for QCD
- There are gauge groups without a sign problem:
  - G2, SU(2), Sp(2N), …



Gattringer, Lang

Spectroscopy

- Spectroscopy:
- $C(n_t) = \left\langle \mathcal{O}(n_t)\mathcal{O}^\dagger(0) \right\rangle = \sum_k A_k e^{-aE_k n_t}$
  - At large times only the ground state survives
- Extraction of higher energy levels:
  - Double-exponential fit, GEVP, …
- Operators specified by quantum number



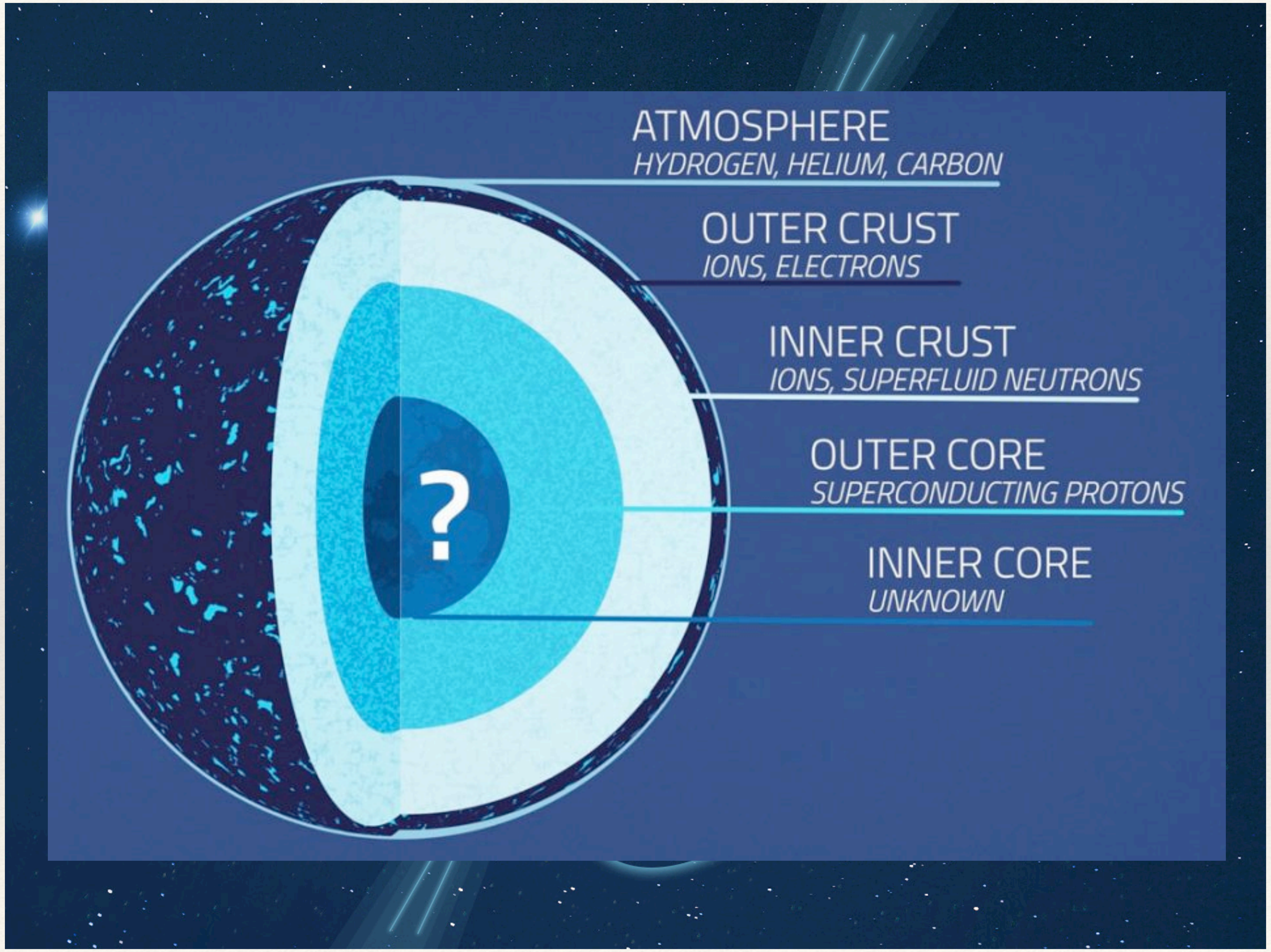
Gattringer, Lang

Using lattice field theory for dark matter

Observable	Application
Equation of state	Dark matter in neutron stars
Scattering: $2 \to 2$ $3 \to 2$	Small Scale Structure Relic density

Neutron Stars

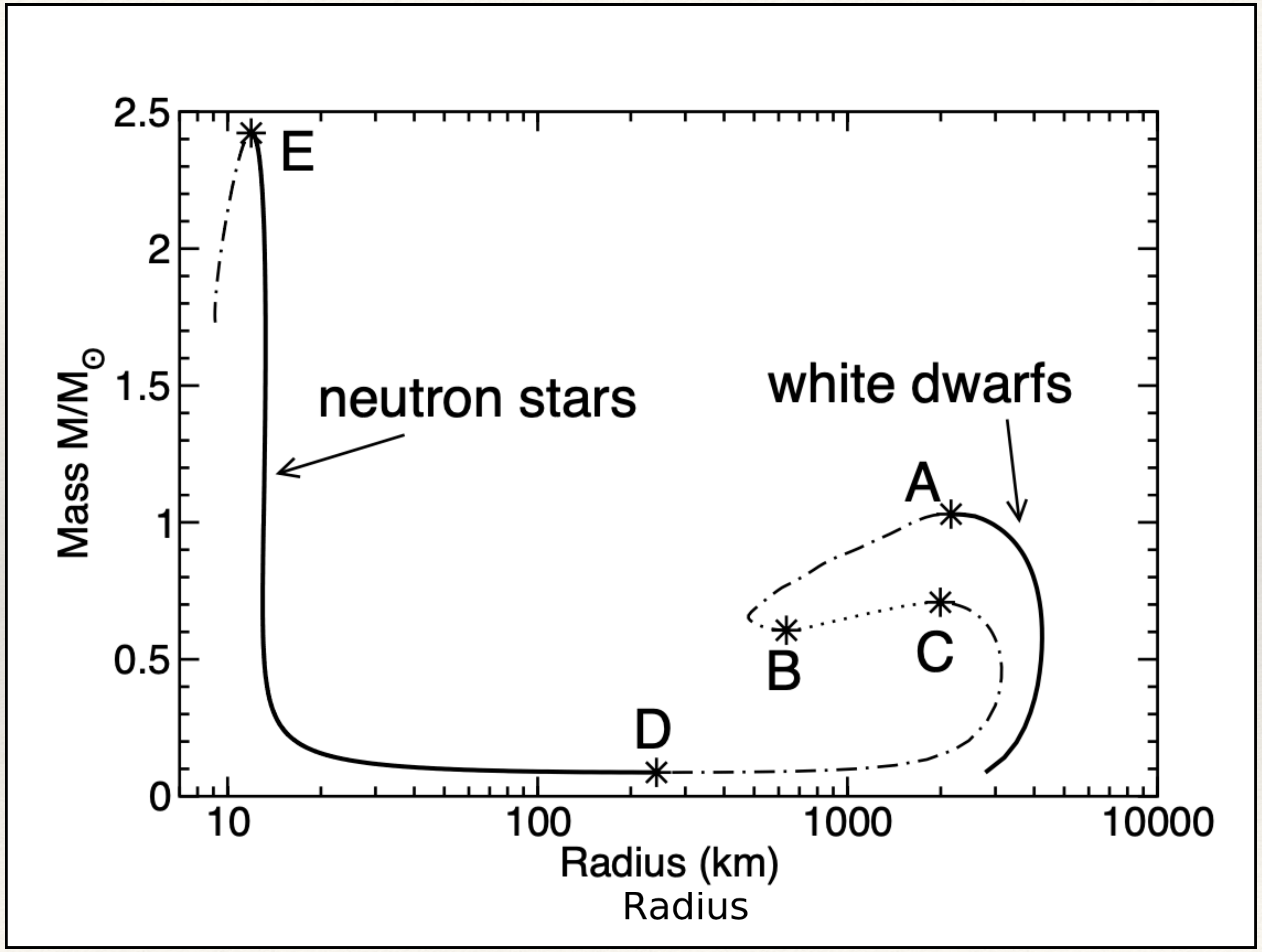
- Created as a remnant of a massive star in a core-collapse super nova
- Super dense with more than 2 $M_{\odot}$ at around 10 km radius (Roughly the size of Ljubljana)
- Interior well understood up to the core
- In the core:
    - Quark matter? Hyperons?



European Space Agency

TOV equation

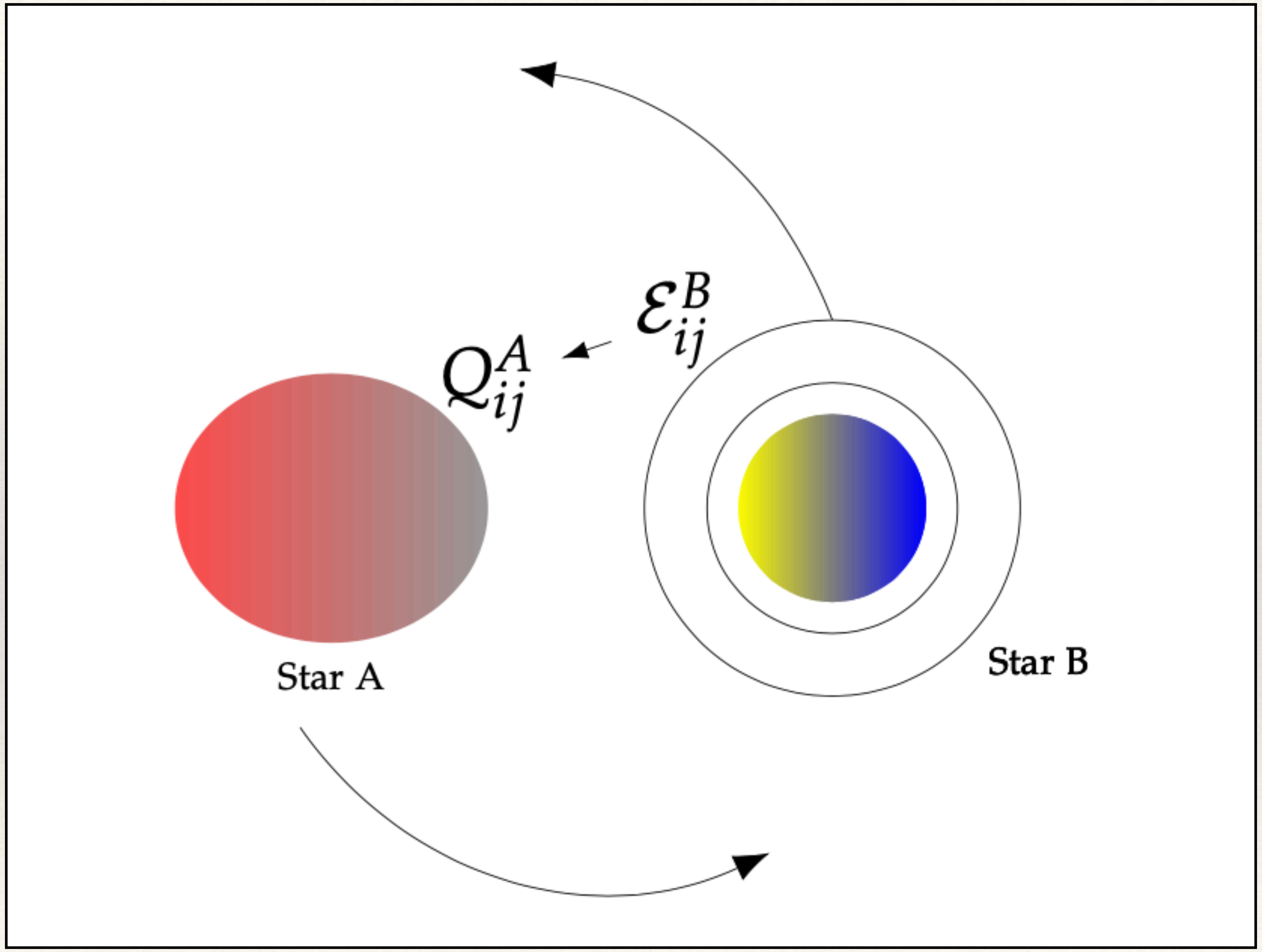
- Mass and radius are obtained from TOV equation
- Input: Equation of state $\epsilon(P)$
- Output: Mass, Radius, …
- Iterate over central pressures
    - Mass-radius relation
- Links the microscopic EoS to macroscopic quantities



Kurkela et al. - ApJ 789 127 (2014)

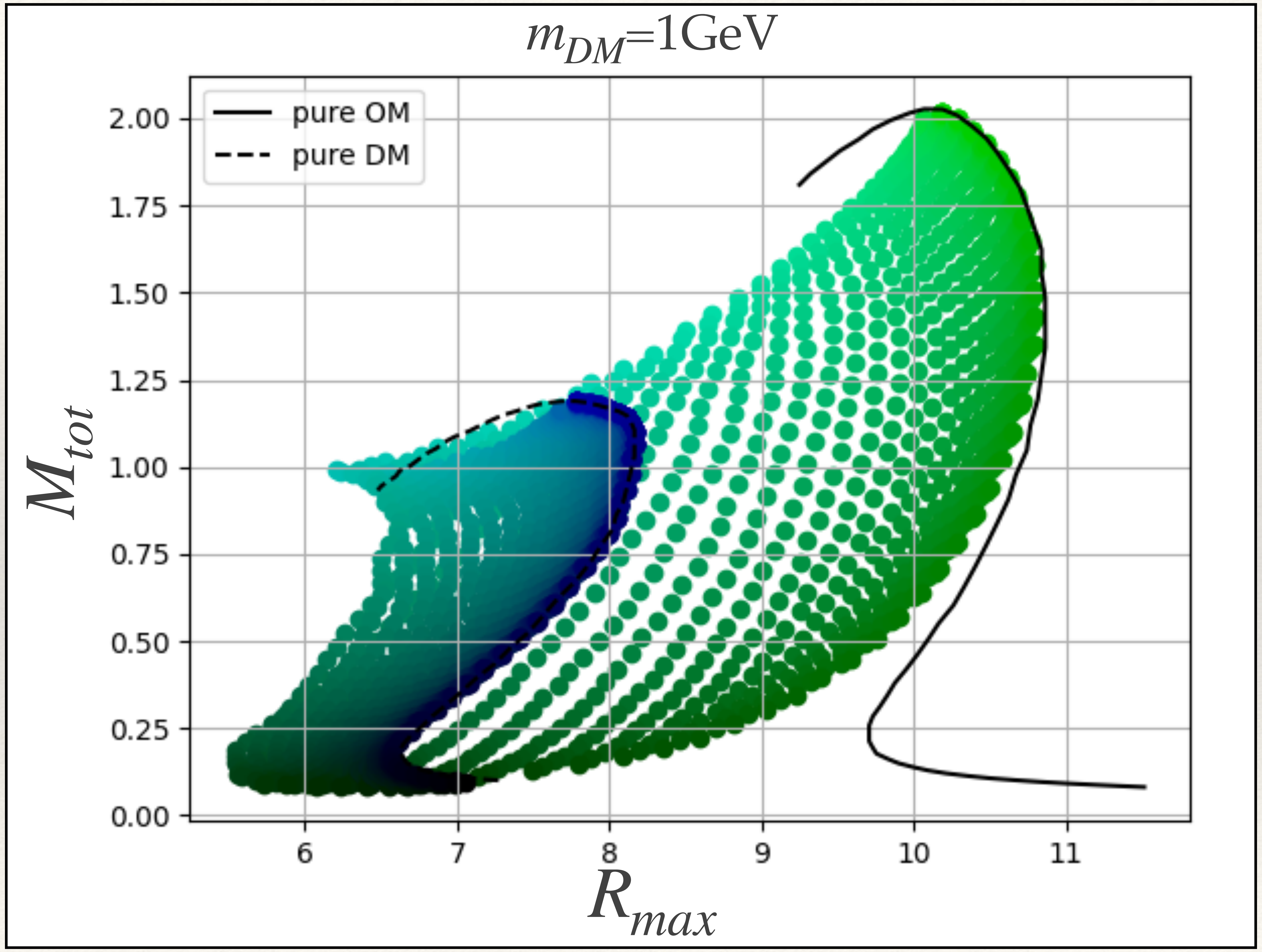
Tidal Deformability

- Tidal field induces a quadrupole deformation
- Can be calculated from simultaneously to the TOV-equations
- Constraint by LIGO (GW170817):
  - $\Lambda < 800$ (@ $1.4\ M_{\odot}$)
- We are in the gravitational wave era!



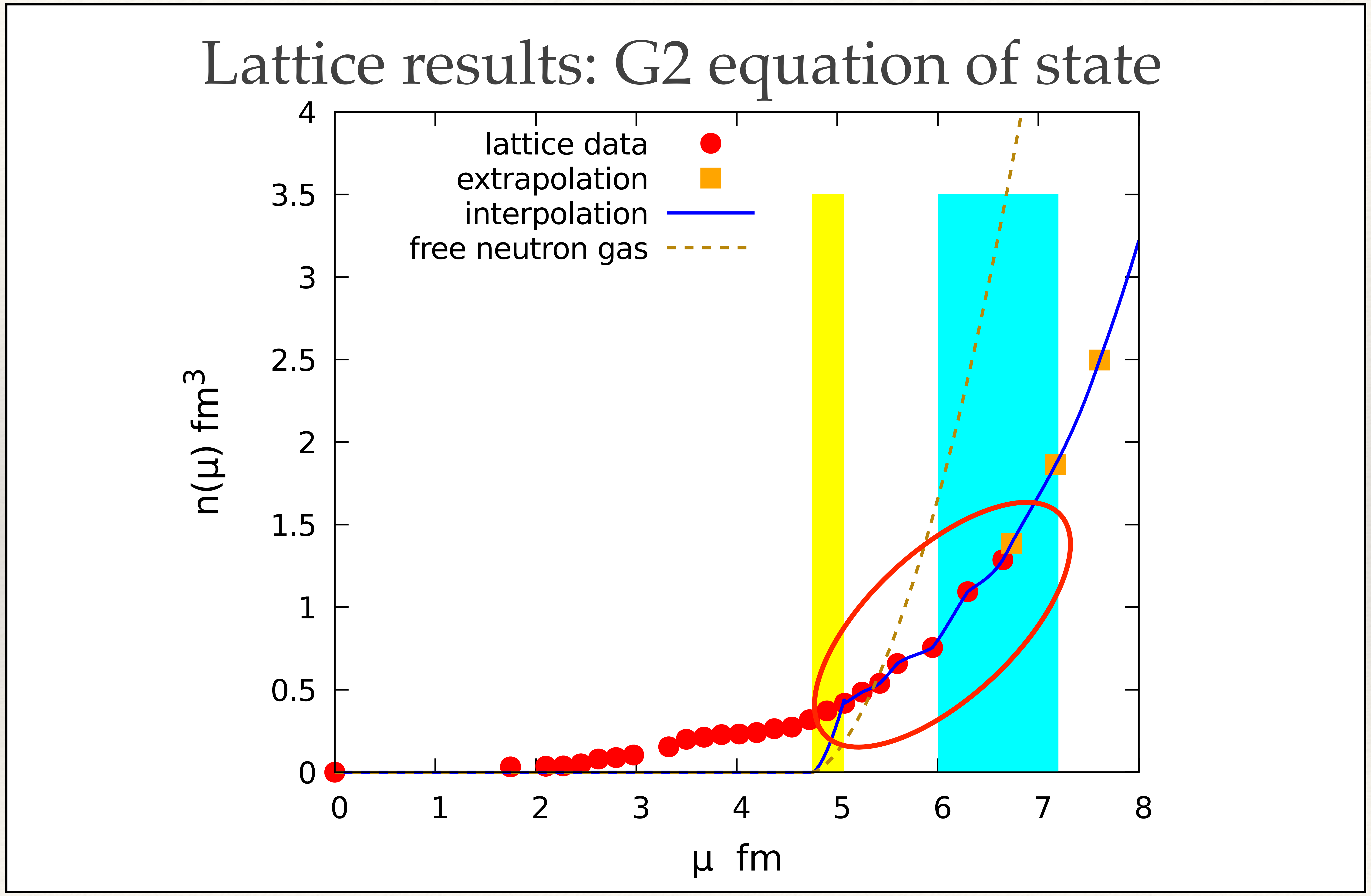
2-fluid TOV equation

- Addition of dark matter:
  - Add a second fluid that only interacts via gravity
- Result:
  - *Dark* halo or core
  - Alters neutron star properties
- Inputs:
  - EoS (OM and DM)
  - $p_{0,OM}$, $p_{0,DM}$



Equations of state

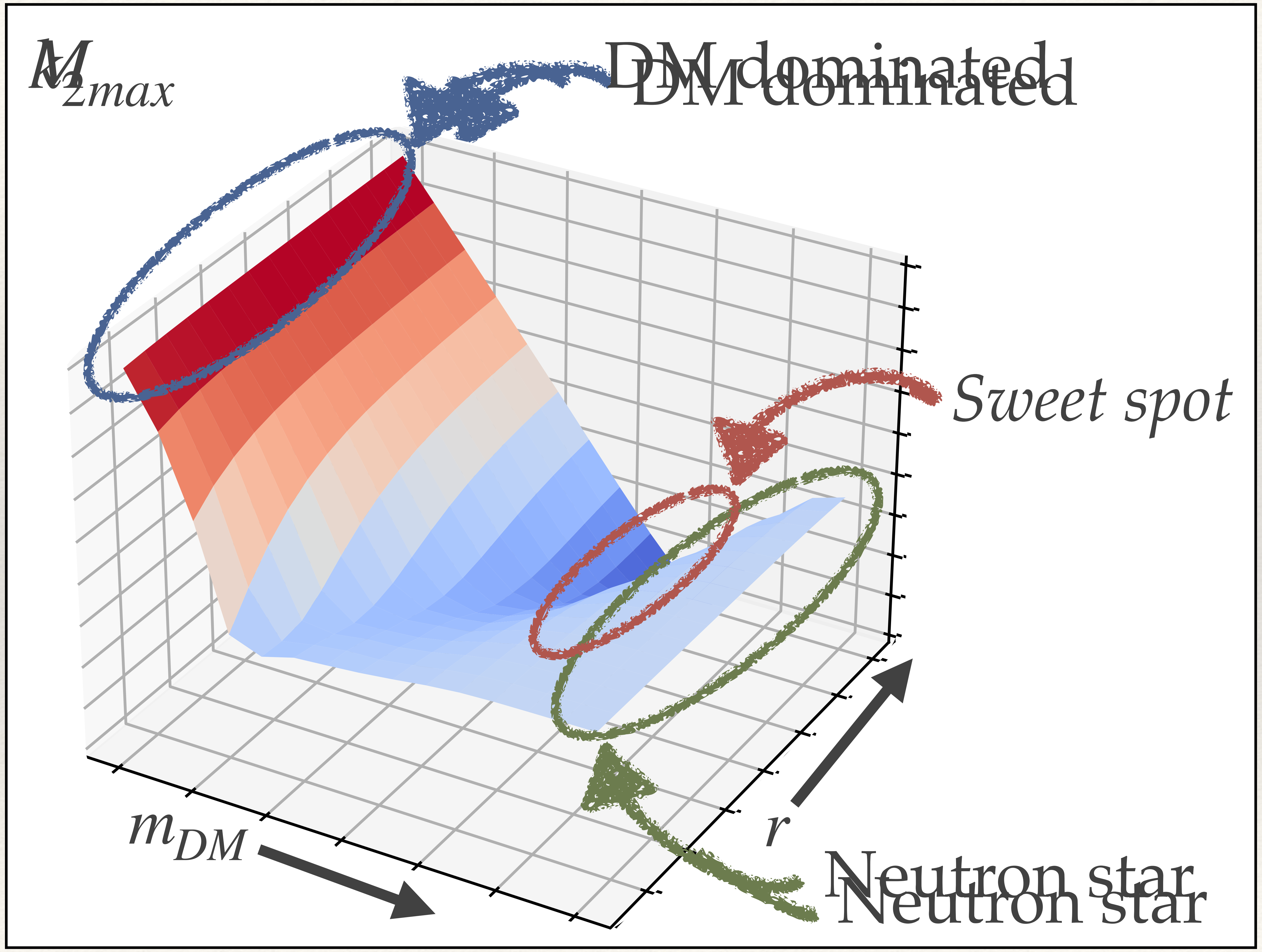
- *Standard* for SM matter
- Finite density result from G2
  - No sign problem
  - DM is lightest fermionic bound state
  - Non-perturbative signatures
- Alternatives:
  - Sp(2N) with different fermions



Hajizadeh, Maas - Eur.Phys.J.A 53 (2017)
Kurkela et al. - ApJ 789 127 (2014)

Preliminary results

- Dark and ordinary matter dominated stars
- *Sweet spot* in between
- Usually lower mass and radius
  - EoS gets *boosted*
- Opens up parameter space
- Similar for the tidal deformability

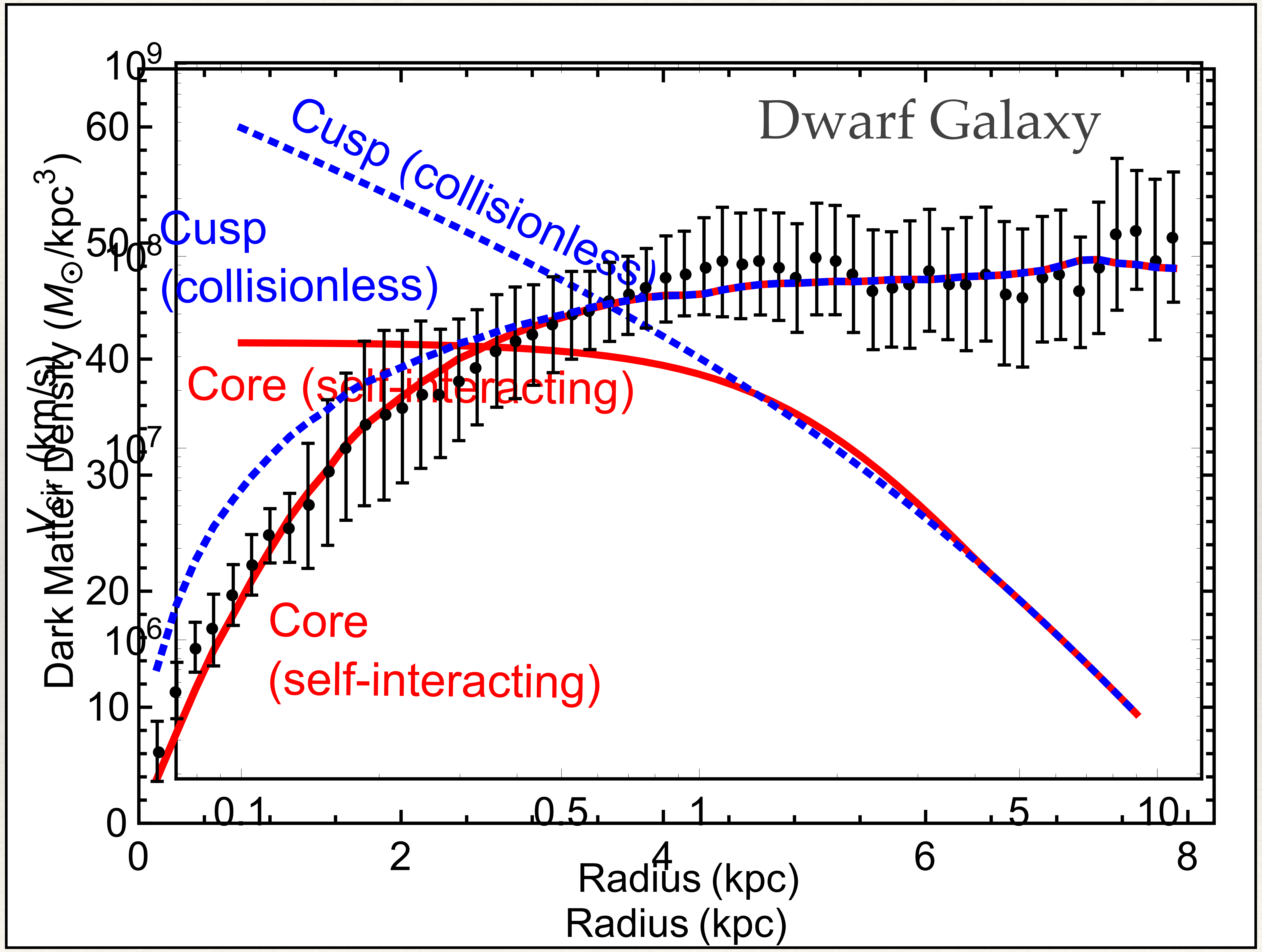


Using lattice field theory for dark matter

Observable	Application
Equation of state	Dark matter in neutron stars
Scattering: $2 \to 2$	Small Scale Structure
$3 \to 2$	Relic density

Self-interaction

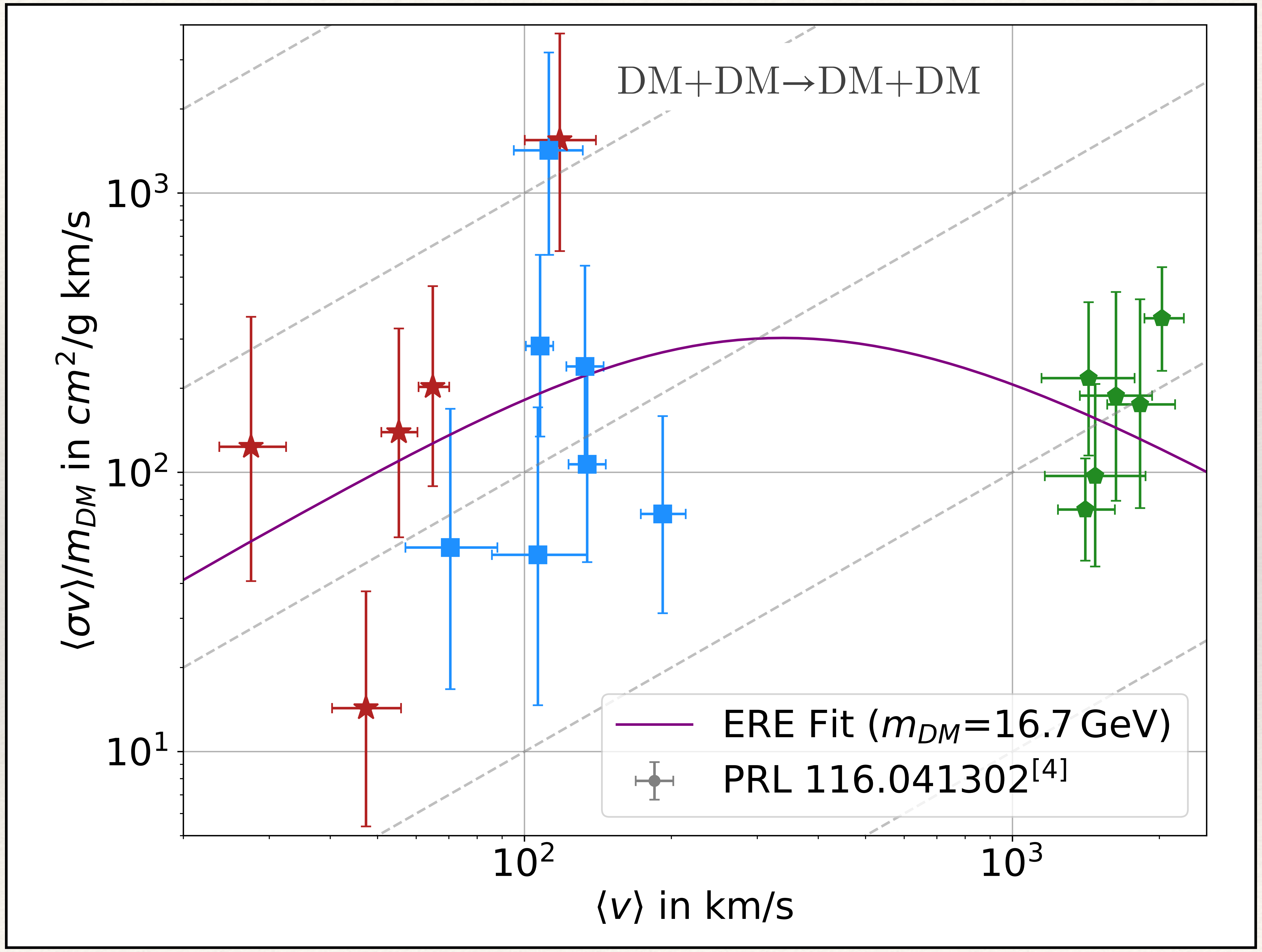
- "Small structure problems"
  - *Diversity, too-big-to-fail, missing satellites, <u>cusp vs. core</u>*
- Core-like shape preferred
  - Hints towards self-interaction
- Upper bounds on cross-section from the bullet cluster



Tulin, Yu: arXiv:1705.02358 (2017)

Velocity-dependent cross-section

- "Dark matter halos as particle colliders"
- Mild velocity dependence @ non-relativistic velocities
- Relies on simulations of dark halos
  - model-dependent

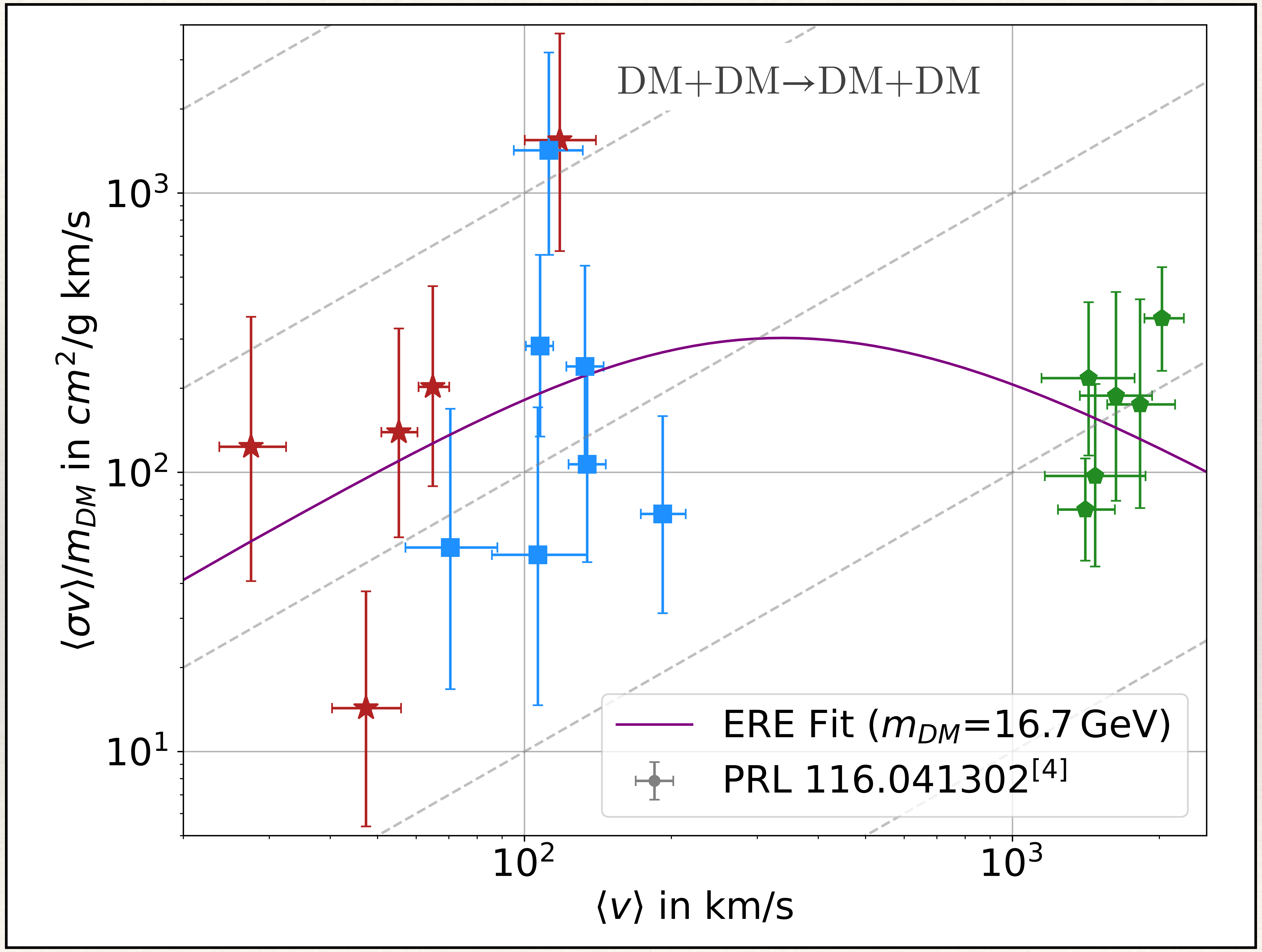


Kaplinghat et al: Phys. Rev. Lett. 116 (2016)

Velocity-dependent cross-section

- DM in halo thermalized
- $\langle \sigma v \rangle = \int_0^{v_{esc}} dv\, \sigma(v)\, v\, f(v)$
  - v - rel. velocity, $f(v)$ - Maxwellian
- Can be done on the lattice
  - $\sigma(v)$ needed

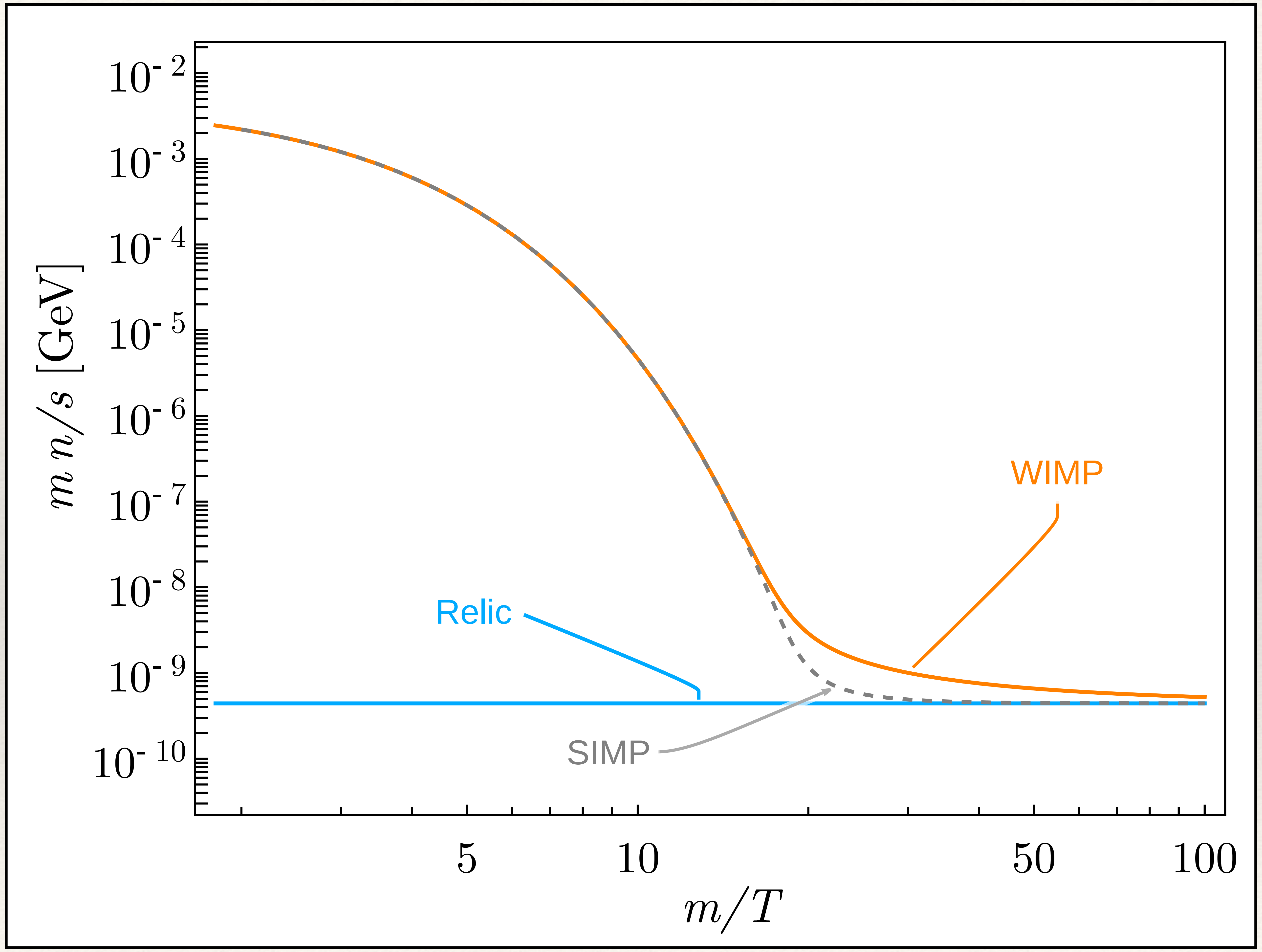
Results later



Kaplinghat et al: Phys. Rev. Lett. 116 (2016)

Relic density

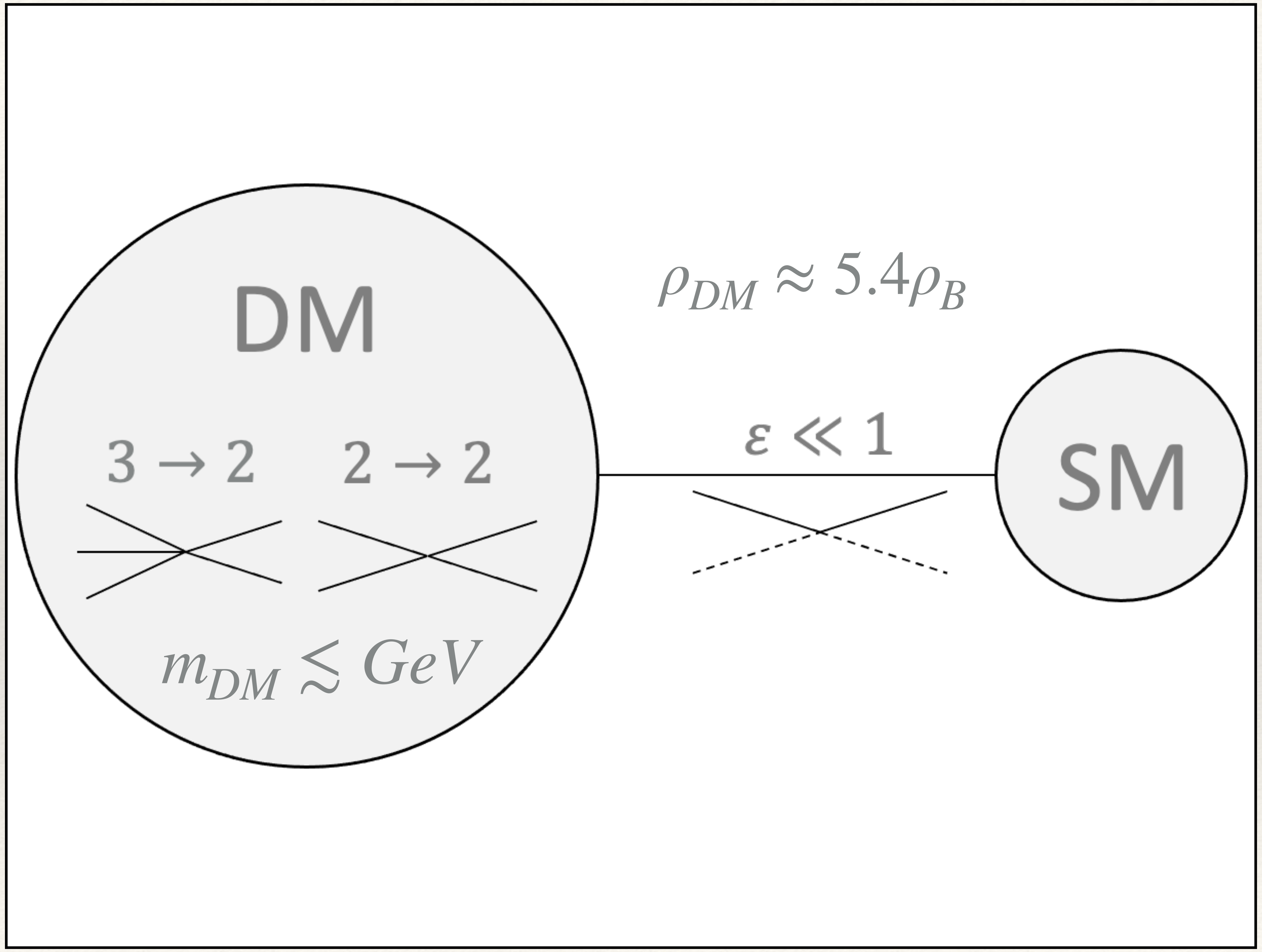
- Possibility: Dark matter as as a thermal relic from the early universe
- Handle on the dark matter abundance
- Solve Boltzmann equations
  - Temperature decreases → interaction "freezes out"
- Example:
  - WIMP: DM + DM → SM + SM



Frumkin et al: Phys.Rev.Lett. 130 (2023)

Strongly Interacting Massive Particles

- Alternative freeze-out paradigm
- Number lowering process in the dark sector
  - Addresses self-interaction
- Coupling to the SM sector needed to prevent heat-up
  - Mediator enables direct detection



Hochberg et al.: Phys. Rev. Lett. 113 (2014)

UV realisation

- Strong coupling arises *naturally* in confining gauge theories
- $\mathscr{L} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \bar{q}_i(i\gamma^\mu D_\mu - m_i)q_i$
- Symmetry depends on representation
  - Fundamental, adjoint, antisymmetric, …
- Also different breaking patterns

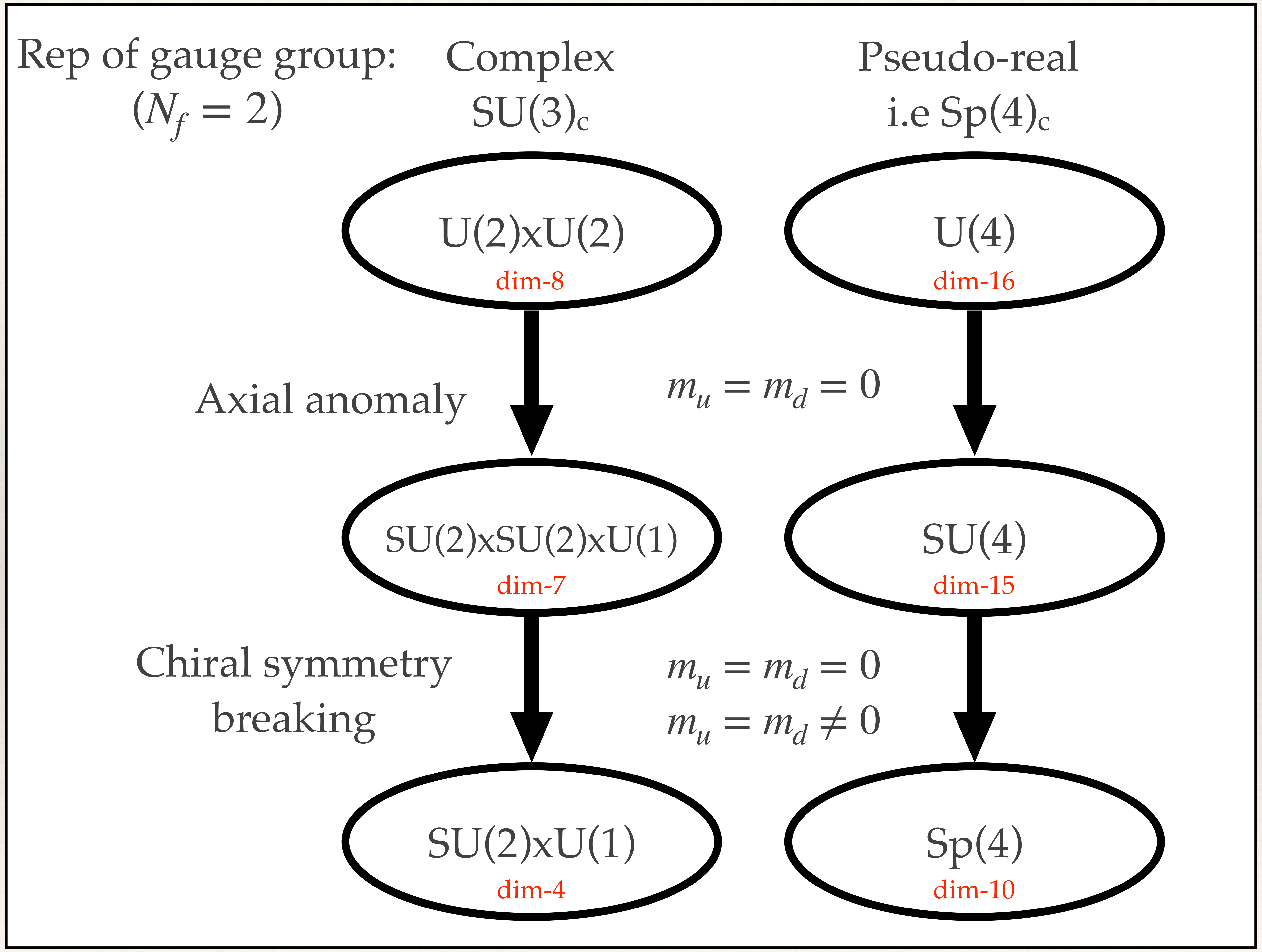
Symmetry of the UV Lagrangian

Representation of gauge group	Flavour symmetry
Complex	U(2)xU(2)
Real	U(4)
Pseudoreal	U(4)

Kulkarni et al.: SciPost Phys. 14 (2023)

Minimal realisation

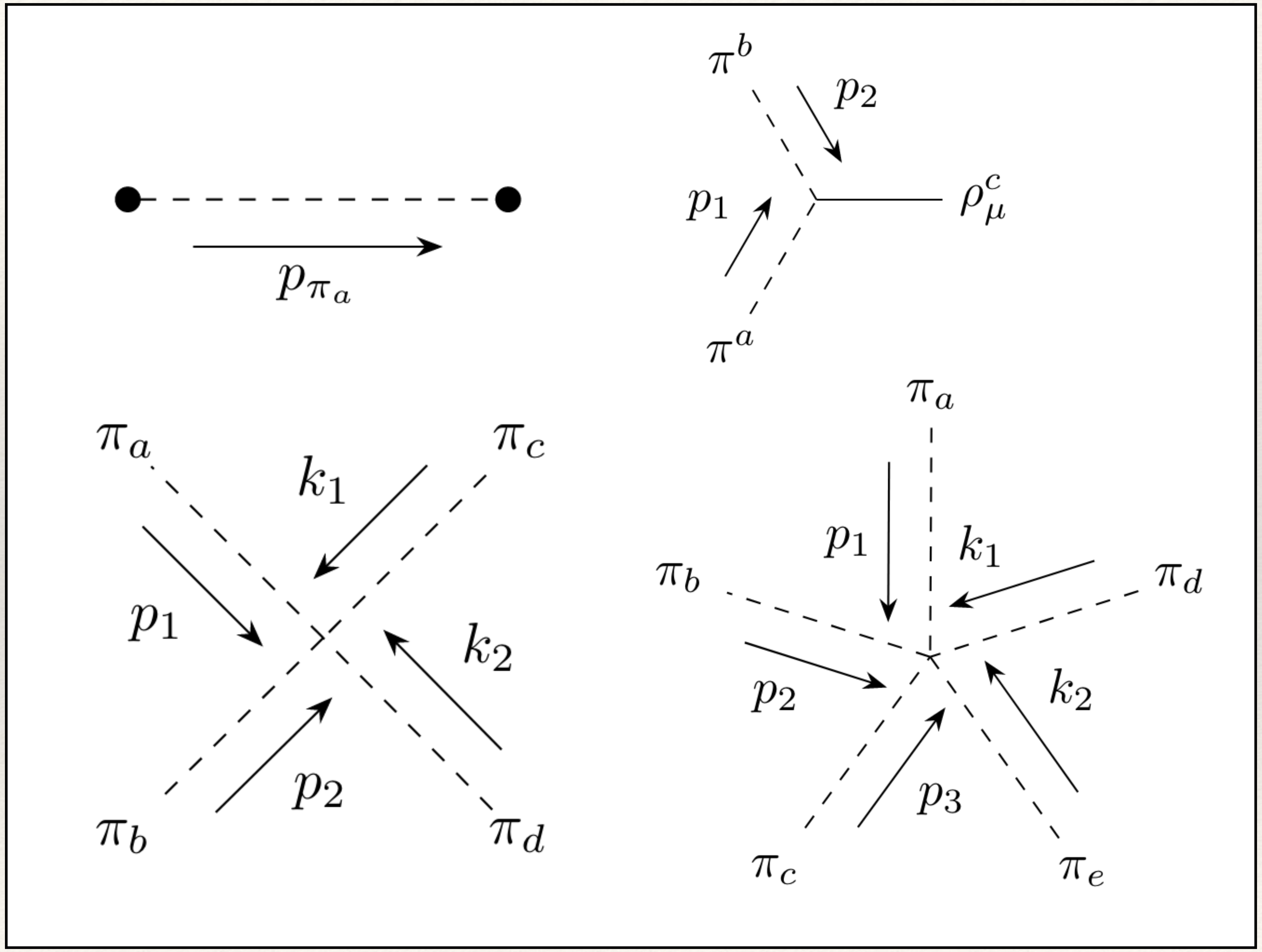
- Pseudo-real rep of gauge group with $N_f = 2$
  - Sp(4) flavour symmetry
- Mixing of left- and right handed components (Weyl-fermions)
  - Symmetry is enlarged
- Result: 5 pNGBs
  - $3 \rightarrow 2$ process possible
  - WZW description in ChPT



Kulkarni et al.: SciPost Phys. 14 (2023)

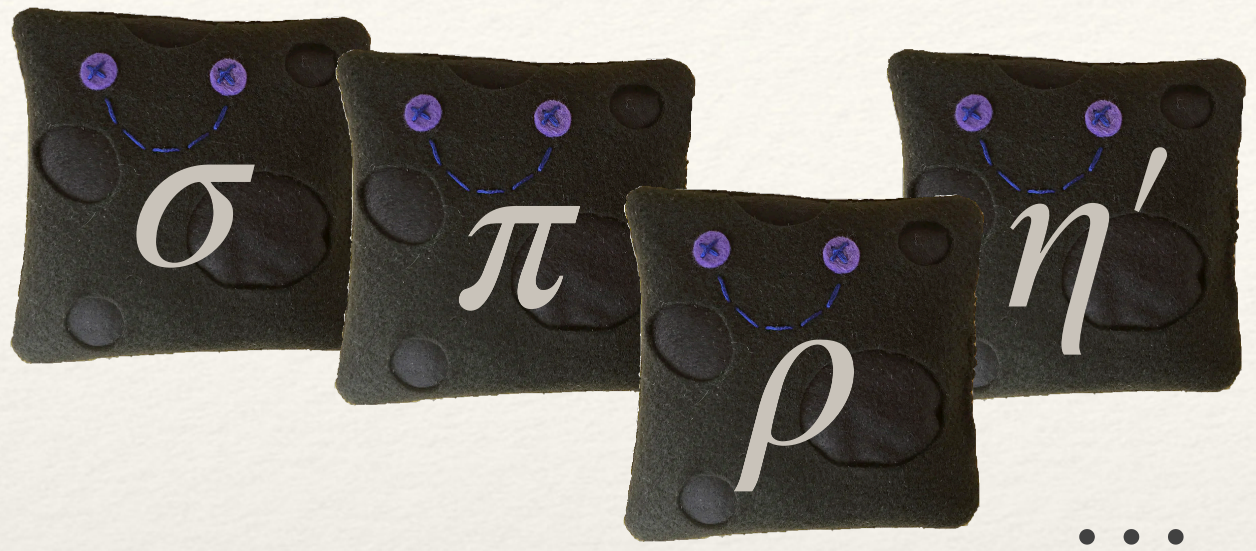
Sp(4) ChPT

- Effective description in terms of 5 *dark Pions*
- Include a vector particle and a mediator to the standard model
- Include 3 → 2 via Wess-Zumino-Witten term
- Relies on low energy constants: Masses, scattering length, …



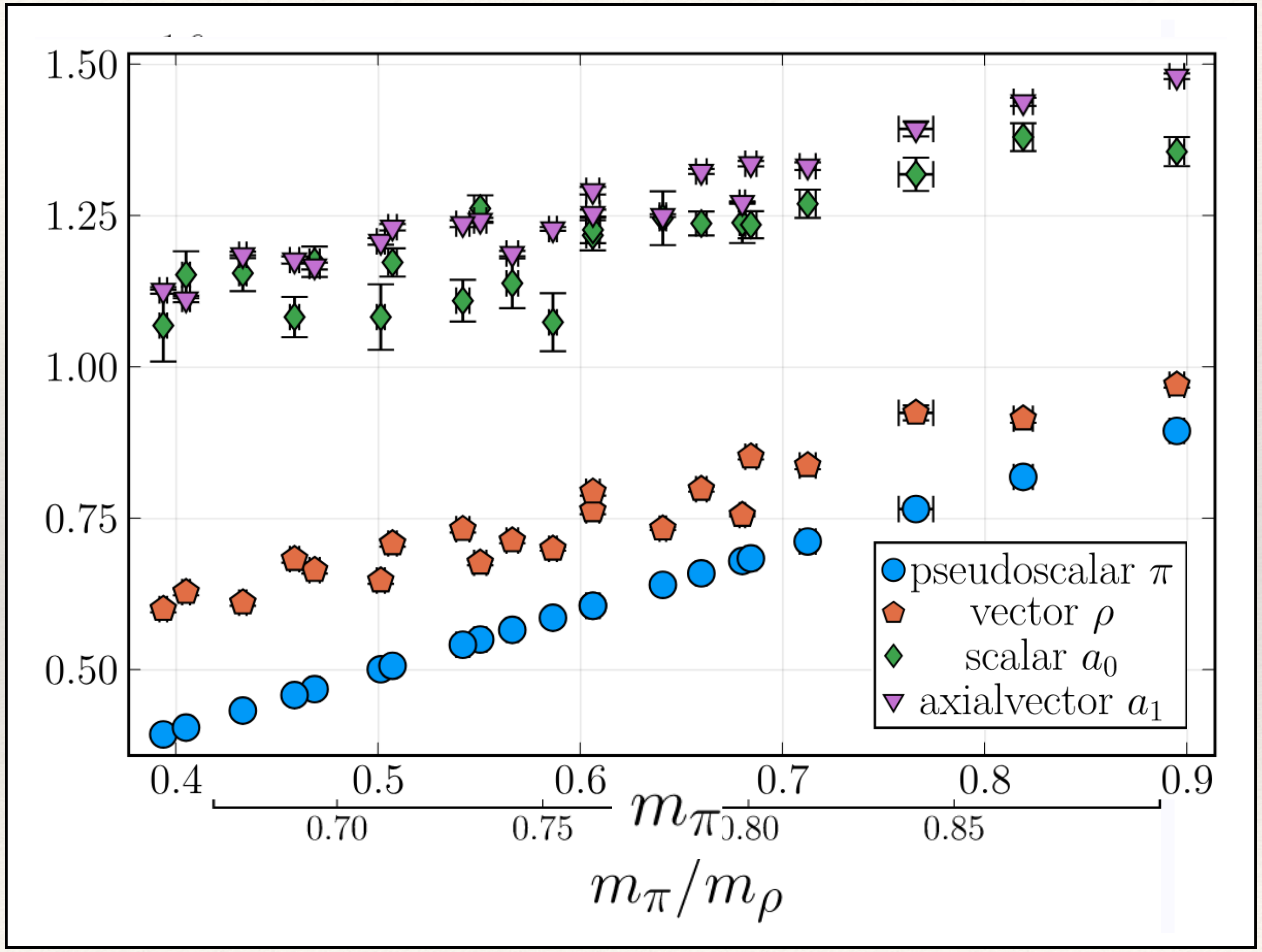
Kulkarni et al.: SciPost Phys. 14 (2023)

Particle phenomenology



…

- Zoo of dark hadrons
- 5 Pions & 10 Rhos lightest non-singlets
- No fermionic bound states
- Light η' relevant for $\pi\pi$ scattering
  - Limits ChPT validity



Bennett et al. - JHEP 12 (2019)
Bennett et al - Phys. Rev. D 109 (2024)

Scattering phenomenology

- Flavour symmetry allows processes which tensor products match

Particle	J^P	Multiplet in Sp(4)
π	0^-	**5**
ρ	1^-	**10**
σ	0^+	**1**
$\dots$		

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 2(35)$$

$\pi\pi \to \pi\pi$ (I=0,1,2)

$\pi\pi \to \rho$ (I=1)

$\pi\pi \to \pi\pi\pi$ (I=1)

$\pi\pi \to \pi\pi\rho$ (I=0,1,2)

etc.

Bennett et al - Phys. Rev. D 109 (2024)
Feger et al.: Comp.Phys.Com 257 (2020)

Scattering phenomenology

- **14**-dim:
  - (Probably) contributes most to $\pi\pi$ -scattering
  - 14 out of 25 possible combinations of Pions

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 2(35)$$

$\pi\pi \rightarrow \pi\pi$ (I=0,1,2)

$\pi\pi \rightarrow \rho$ (I=1)

$\pi\pi \rightarrow \pi\pi\pi$ (I=1)

$\pi\pi \rightarrow \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Scattering phenomenology

- **1**-dim:
  - (Probably) no large contribution to $\pi\pi$ -scattering
  - Mixes in other scattering channel
  - Numerically challenging

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 2(35)$$

$\pi\pi \rightarrow \pi\pi$ (I=0,1,2)

$\pi\pi \rightarrow \rho$ (I=1)

$\pi\pi \rightarrow \pi\pi\pi$ (I=1)

$\pi\pi \rightarrow \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Scattering phenomenology

- **10**-dim:
  - Mixing with the Rho
  - $\pi\pi\pi \to \pi\pi$
- Work in progress

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 2(35)$$

$\pi\pi \to \pi\pi$ (I=0,1,2)

$\pi\pi \to \rho$ (I=1)

$\pi\pi \to \pi\pi\pi$ (I=1)

$\pi\pi \to \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Phenomenology of scattering channels

Done ✅ Results in this talk

- **14-dim:**
  - Makes up most $\pi\pi$ scattering (14/25)
  - Easiest on the lattice
- **10-dim:**

  Work in progress
  - Mixing with dark ρ
  - $\pi\pi\pi \to \pi\pi$
- **1-dim:**
  - Mixing with other states

$Sp(4)_f$

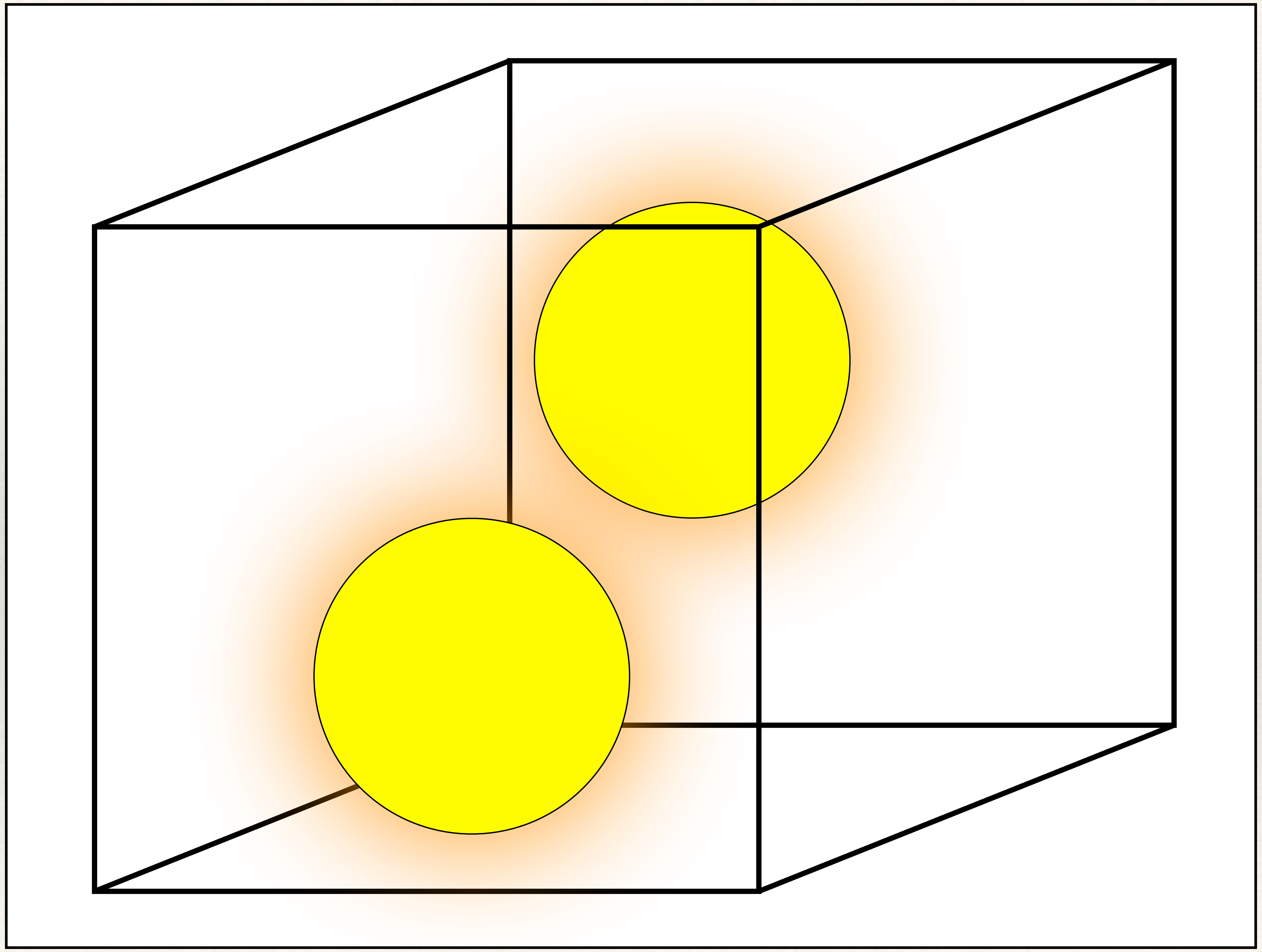
$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$
$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$
$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 2(35)$$
$$\pi\pi \to \pi\pi \text{ (I=0,1,2)}$$
$$\pi\pi \to \rho \text{ (I=1)}$$
$$\pi\pi \to \pi\pi\pi \text{ (I=1)}$$
$$\pi\pi \to \pi\pi\rho \text{ (I=0,1,2)}$$

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Scattering on the lattice

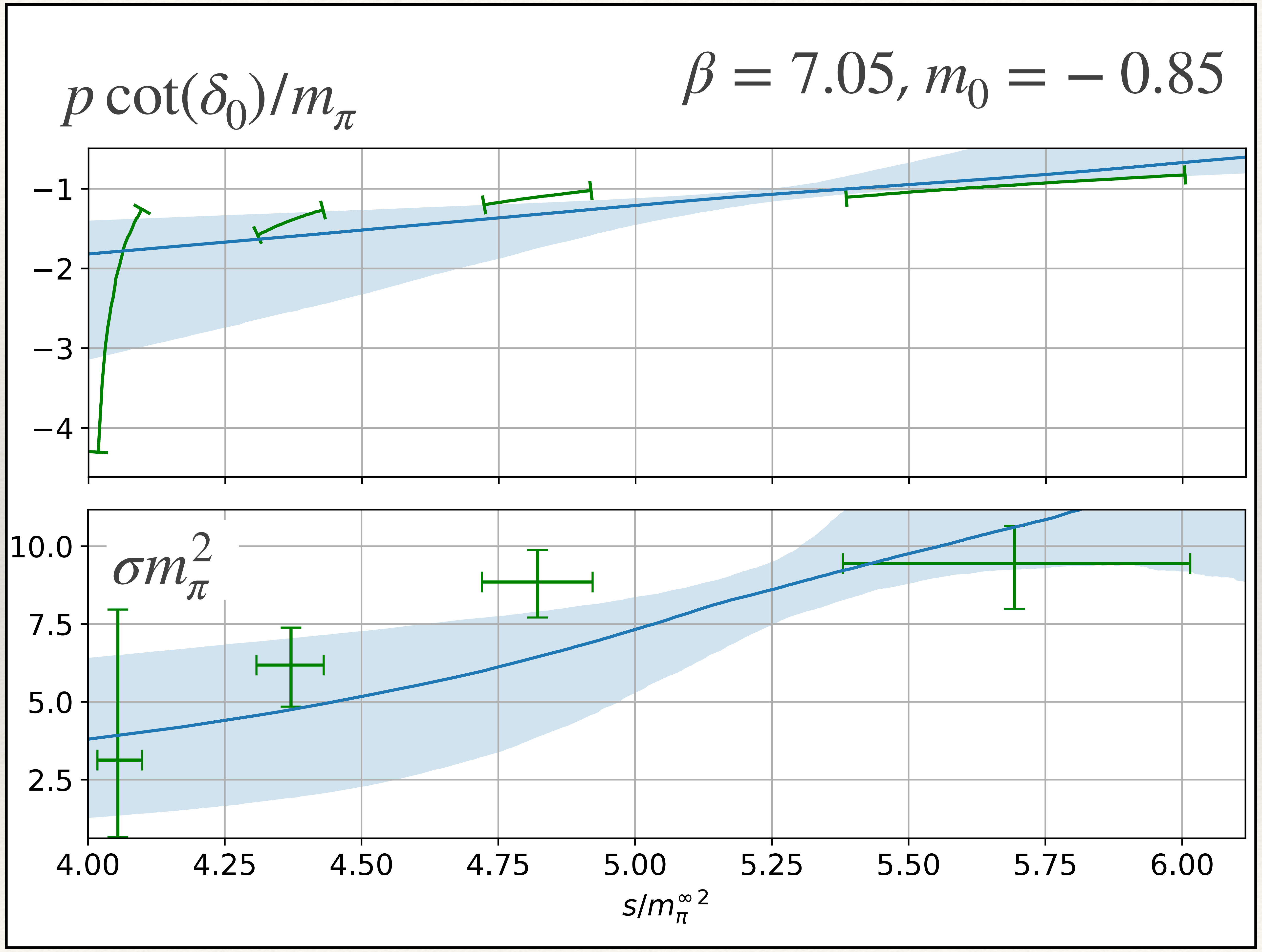
- Relate finite volume energy levels with infinite volume scattering properties
  - "Lüscher quantization condition"
- $\tan(\delta(\sqrt{s})) = f(E, \overrightarrow{P}, L)\,|_{E=E(L)}$
- Result: Energy-dependent phase-shift



Lüscher et al.: Commun. Math. Phys. 104/105 (1986)
Briceño et al.: Rev.Mod.Phys. 90 (2018) 2, 025001

Phase shift

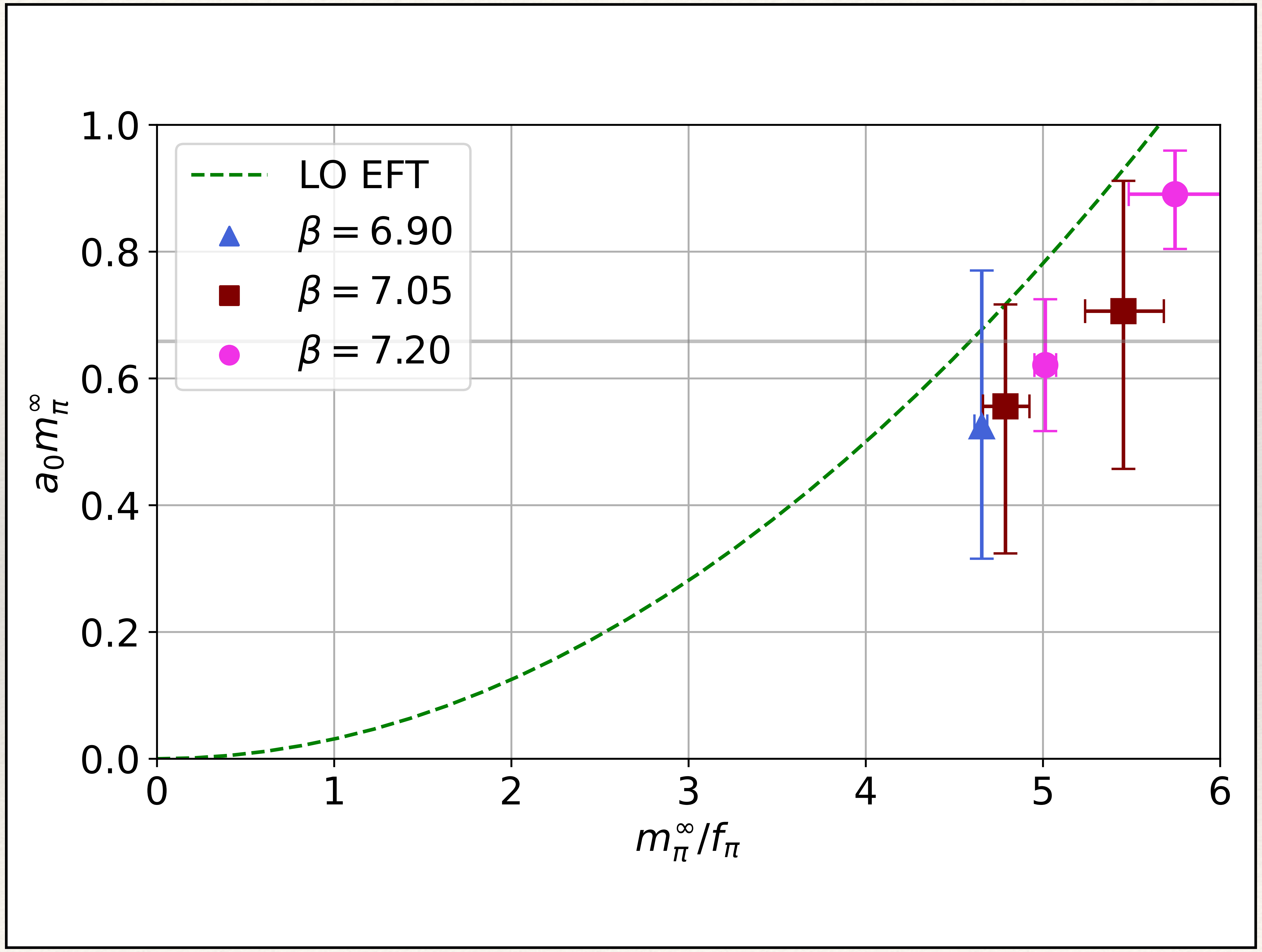
- Effective range expansion:
  - Expand phase shift in $\mathcal{O}(p^2)$
- Different parameterizations possible
- Access to $\sigma(s)$
  - Relative velocity $v(s)$



YD, F. Zierler and A. Maas: arXiv:2405.06506

χ-pT comparison

- Prediction: $a_0 m_\pi = \dfrac{1}{32}\left(\dfrac{m_\pi}{f_\pi}\right)^2$
- Potential systematics
- Promising for ChPT
  - NLO?

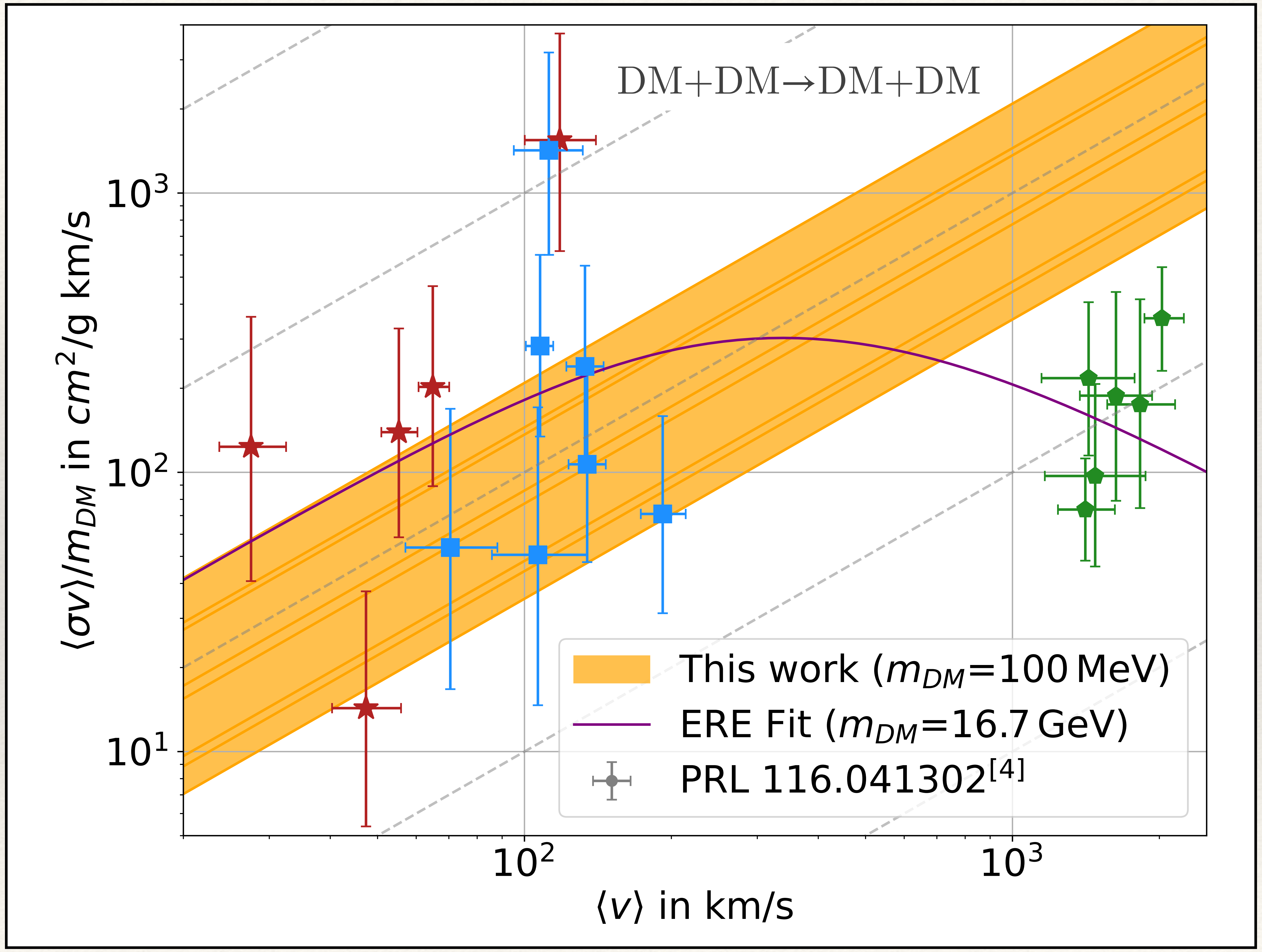


YD, F. Zierler and A. Maas: arXiv:2405.06506
Bijnens and Lu: JHEP, 03:028, (2011)

Velocity-weighted cross-section

$$\langle \sigma v \rangle = \int_0^{v_{esc}} dv\, \sigma(v)\, v\, f(v)$$

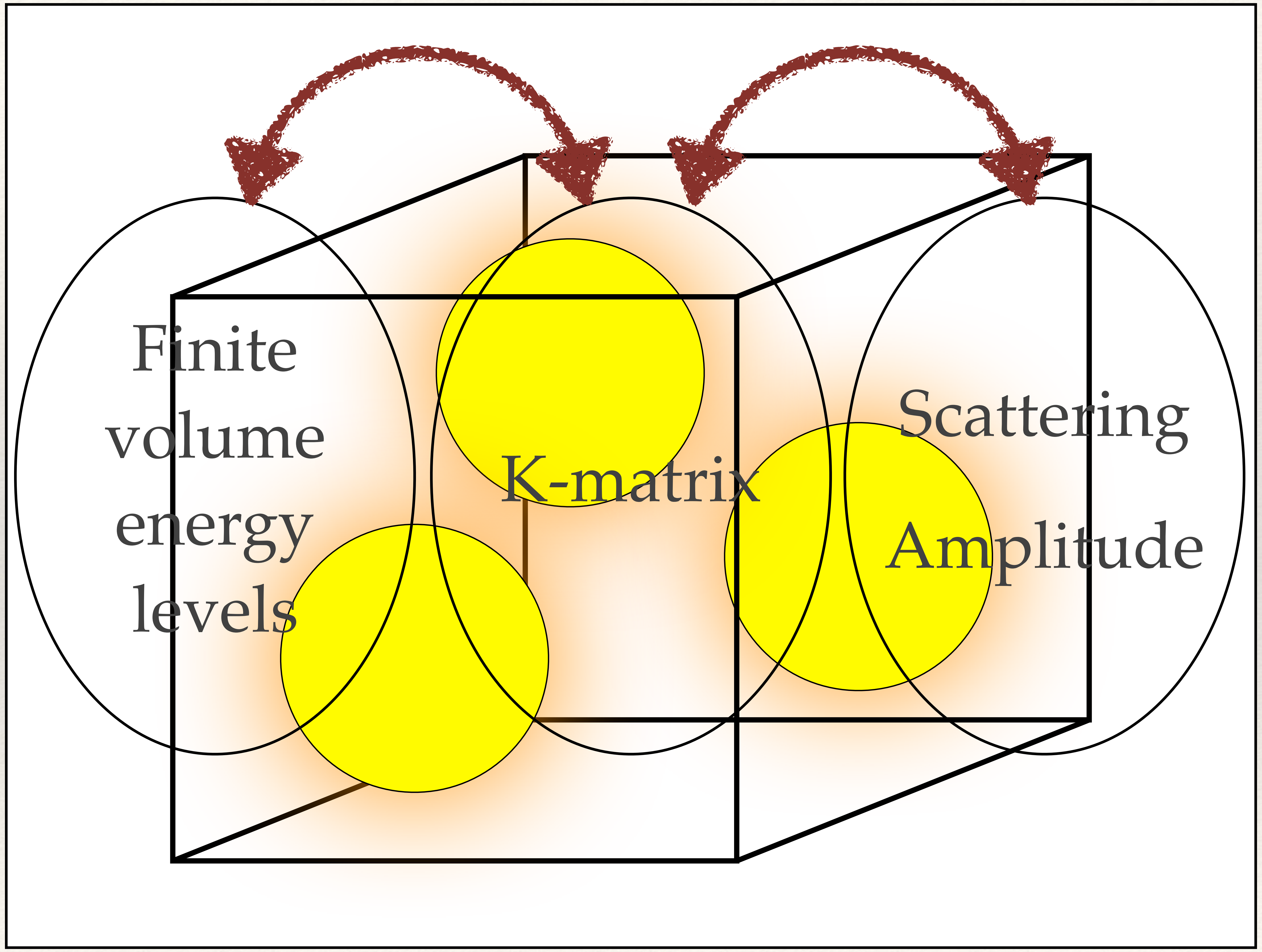
- Assumption: s-wave and maximal scattering channel
- No sign for a velocity dependence
  - Discrepancy in $a_0 m_{DM}$
- $m_{DM} \sim 100\,\text{MeV}$ predicted by SIMP
- Sp(4) not ruled out



YD, F. Zierler and A. Maas: arXiv:2405.06506
Kaplinghat et al: Phys. Rev. Lett. 116 (2016)

3 particle scattering

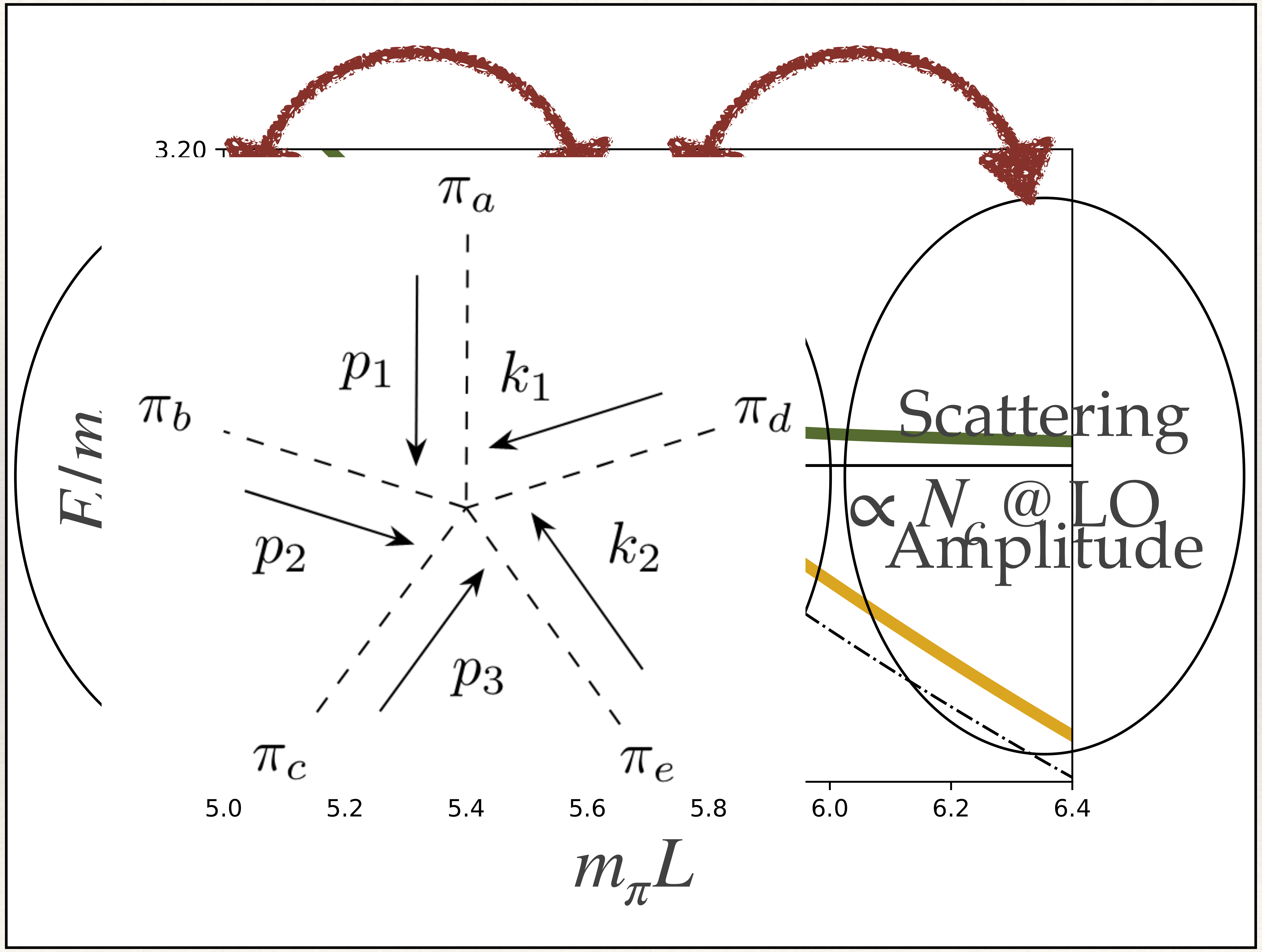
- Extension of finite-volume formalism
- $\det\left[F_3^{-1} + K_3\right] = 0$
- Obtaining energy levels is harder
- Integral equations to relate K to $\mathcal{M}$
- First lattice calculations on $\pi\pi\pi$-scattering only recently achieved



Hansen, Sharpe - Phys. Rev. D 90 (2014)
Athari Alotaibi (Poster)

3 particle scattering

- Parametrize infinite volume scattering
  - What would the finite volume energy levels look like?
- $\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{pmatrix}\right] = 0$
  - K_{23} parametrizes $\pi\pi\pi \to \pi\pi$
- Sp(4): Translate ChPT prediction from the WZW term to K_{23}
- Framework can be used with lattice data



Hansen, Sharpe - Phys. Rev. D 90 (2014)
Athari Alotaibi (Poster)

Summary & Outlook

- Lots of interesting applications for the lattice in dark matter physics
- First principle verification for low energy constants important
- With $\pi\pi \rightarrow \rho$ & $\pi\pi\pi \rightarrow \pi\pi$ we will obtain a good understanding of the model

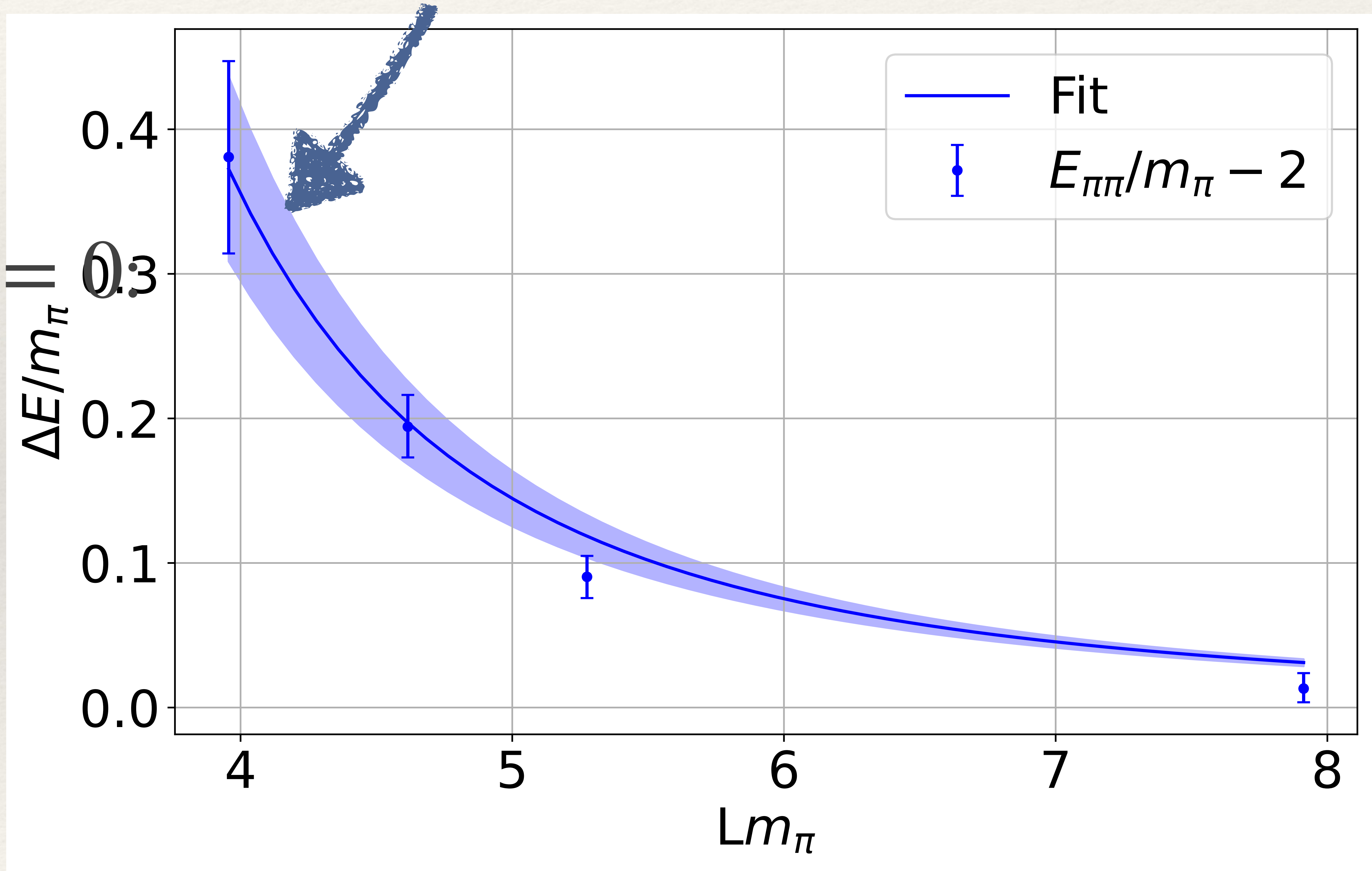
Thank you!

Collaborators: Fabian Zierler, Axel Maas,
Kevin Radl, Suchita Kulkarni, Max Hansen

Energy levels

- Power-like finite volume effects:
- Expansion of Lüscher formula for $\vec{P} = 0$:
- $\Delta E = E - 2m_\pi = -\frac{4\pi a_0}{m_\pi L^3}\left(1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2}\right)$
- Full function f gives access to $\delta(E_{cm})$

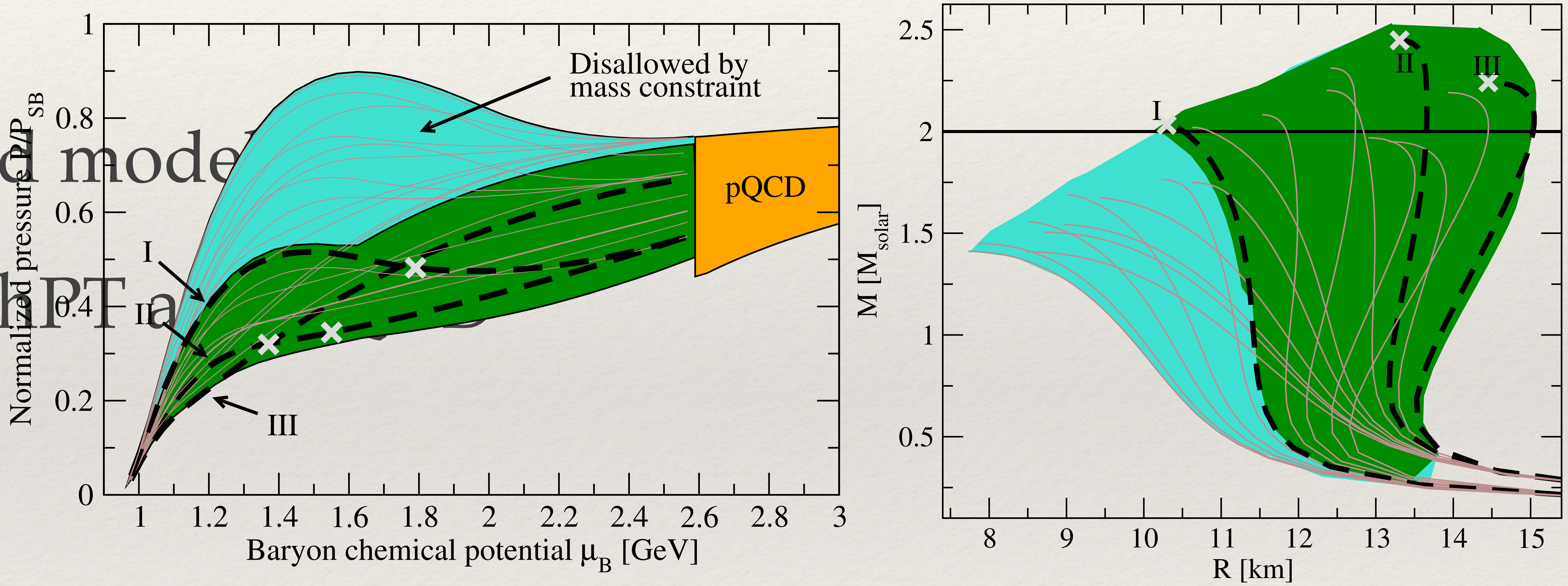
Slope contains scattering length



Jenny et al.: Phys. Rev. D 105 (2022)

OM Equations of state

- Cover a large range of standard mode
- Interpolate between nuclear ChPT a
- Apply constraints



Kurkela et al. - ApJ 789 127 (2014)

Result tables

β	am_0	N_L	N_T	n_{config}	m_π/m_ρ	am_π	$aE_{\pi\pi}$	af_π	$\langle P \rangle$
6.9	-0.87	10	20	976	0.8744(43)	0.7425(12)	1.4961(22)	0.1313(25)	0.550680(46)
6.9	-0.87	12	24	400	0.8754(41)	0.7414(15)	1.4891(27)	0.13195(91)	0.550441(54)
6.9	-0.87	16	32	100	0.8762(28)	0.74060(96)	1.4820(23)	0.1324(35)	0.550525(61)
6.9	-0.9	8	16	651	0.795(11)	0.6241(25)	1.2799(48)	0.1014(44)	0.557959(99)
6.9	-0.9	8	24	402	0.811(12)	0.6267(26)	1.2806(53)	0.0969(20)	0.55796(10)
6.9	-0.9	10	20	1273	0.7998(38)	0.5738(12)	1.1602(26)	0.1044(21)	0.557172(40)
6.9	-0.9	12	24	2904	0.8110(22)	0.56409(54)	1.1339(14)	0.10484(87)	0.557009(18)
6.9	-0.9	14	24	942	0.8115(27)	0.56222(63)	1.1280(16)	0.10599(58)	0.556981(26)
6.9	-0.9	16	32	546	0.8156(28)	0.56275(57)	1.1283(12)	0.1064(13)	0.556921(25)
6.9	-0.9	18	36	356	0.8135(24)	0.56121(58)	1.1245(12)	0.10576(91)	0.556987(24)
6.9	-0.91	12	24	1268	0.7698(77)	0.4920(10)	0.9950(27)	0.0949(13)	0.559351(28)
6.9	-0.91	14	24	513	0.7756(81)	0.4857(12)	0.9781(29)	0.0945(23)	0.559409(34)
6.9	-0.91	16	32	435	0.7658(65)	0.48610(86)	0.9765(19)	0.0948(23)	0.559353(27)
6.9	-0.92	12	24	63	0.738(61)	0.416(19)	0.885(14)	0.0853(68)	0.56145(14)
6.9	-0.92	14	24	550	0.699(10)	0.3926(14)	0.7914(40)	0.0724(19)	0.562096(34)
6.9	-0.92	16	32	176	0.670(11)	0.3894(14)	0.7848(35)	0.0821(15)	0.562116(42)
6.9	-0.92	24	32	467	0.7035(31)	0.38649(51)	0.7734(12)	0.08260(35)	0.562077(14)
7.05	-0.835	8	24	402	0.777(13)	0.6585(60)	1.345(15)	0.0481(47)	0.577085(74)
7.05	-0.835	12	24	313	0.790(11)	0.4616(15)	0.9424(32)	0.0792(14)	0.575237(39)
7.05	-0.835	14	24	619	0.7877(91)	0.4417(17)	0.9085(30)	0.0793(20)	0.575368(25)
7.05	-0.835	20	36	100	0.7945(61)	0.4380(10)	0.8792(27)	0.0796(31)	0.575269(29)
7.05	-0.85	12	24	84	0.611(33)	0.3778(57)	0.786(22)	0.0582(39)	0.577835(69)
7.05	-0.85	14	24	167	0.716(26)	0.3496(25)	0.7236(67)	0.0675(17)	0.577429(44)
7.05	-0.85	16	32	101	0.660(17)	0.3375(17)	0.6892(41)	0.0669(11)	0.577413(41)
7.05	-0.85	24	36	100	0.7118(64)	0.33076(97)	0.6638(23)	0.0684(19)	0.577371(24)
7.2	-0.78	8	24	401	0.8770(81)	0.8089(42)	1.617(11)	0.0402(50)	0.590527(59)
7.2	-0.78	10	20	195	0.648(16)	0.5508(48)	1.1345(88)	0.0497(20)	0.589788(65)
7.2	-0.78	12	24	150	0.835(19)	0.4382(34)	0.9024(84)	0.0569(14)	0.589547(56)
7.2	-0.78	14	24	425	0.7762(83)	0.3857(14)	0.7951(35)	0.06569(73)	0.589362(26)
7.2	-0.78	16	32	265	0.7930(90)	0.3809(11)	0.7703(31)	0.0645(11)	0.589253(22)
7.2	-0.78	24	36	508	0.7852(30)	0.36963(39)	0.74360(79)	0.0646(26)	0.5892779(85)
7.2	-0.794	12	24	101	0.732(26)	0.3932(63)	0.823(13)	0.0389(24)	0.590837(54)
7.2	-0.794	14	24	234	0.691(31)	0.3234(26)	0.6888(66)	0.0533(14)	0.590422(39)
7.2	-0.794	16	32	101	0.796(27)	0.3097(17)	0.6463(50)	0.0570(13)	0.590330(40)
7.2	-0.794	28	36	504	0.7163(57)	0.28524(35)	0.57582(97)	0.05689(71)	0.5904516(67)

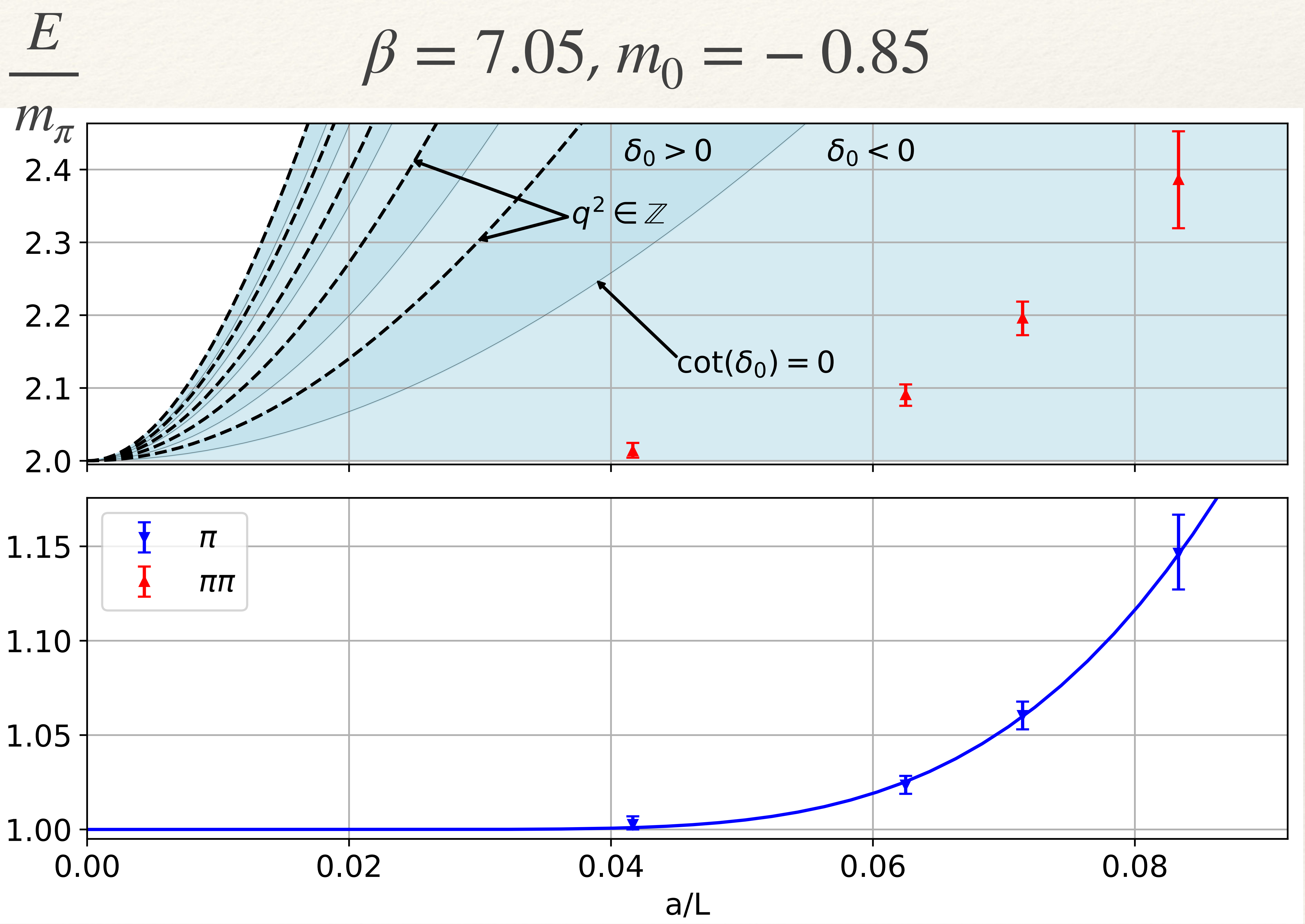
Energy levels

Effective range expansion

β	am_0	$m_\pi^\infty \times 10^4$	$a_0 m_\pi$	$r_0 m_\pi$
6.9	-0.87	7401^{+9}_{-9}	$0.39^{+0.35}_{-0.25}$	59^{+329}_{-110}
6.9	-0.9	5608^{+4}_{-4}	$0.45^{+0.06}_{-0.07}$	$9.3^{+3.5}_{-2.2}$
6.9	-0.91	4845^{+9}_{-9}	$0.36^{+0.12}_{-0.12}$	42^{+57}_{-27}
6.9	-0.92	3845^{+18}_{-31}	$0.52^{+0.23}_{-0.21}$	$6.7^{+9.5}_{-3.8}$
7.05	-0.835	4373^{+9}_{-9}	$0.70^{+0.12}_{-0.21}$	$1.9^{+1.1}_{-0.4}$
7.05	-0.85	3297^{+11}_{-13}	$0.60^{+0.14}_{-0.25}$	$3.7^{+7.4}_{-1.7}$
7.2	-0.78	3696^{+4}_{-4}	$0.80^{+0.06}_{-0.08}$	$2.0^{+0.3}_{-0.2}$
7.2	-0.794	2837^{+12}_{-14}	$0.84^{+0.13}_{-0.16}$	$1.3^{+0.7}_{-0.4}$

Energy levels

- Infinite volume pion mass
- $q = \frac{L}{2\pi}P, \tan(\delta) = \frac{\pi^{\frac{3}{2}}q}{\mathcal{Z}^{\vec{0}}_{00}(1,q^2)}$
  - Non-interacting levels: $q^2 \in \{1,2,..$
  - Resonances: $\mathcal{Z}(1,q^2) = 0$
- One to one mapping of $E_{\pi\pi}(L)$ to sign

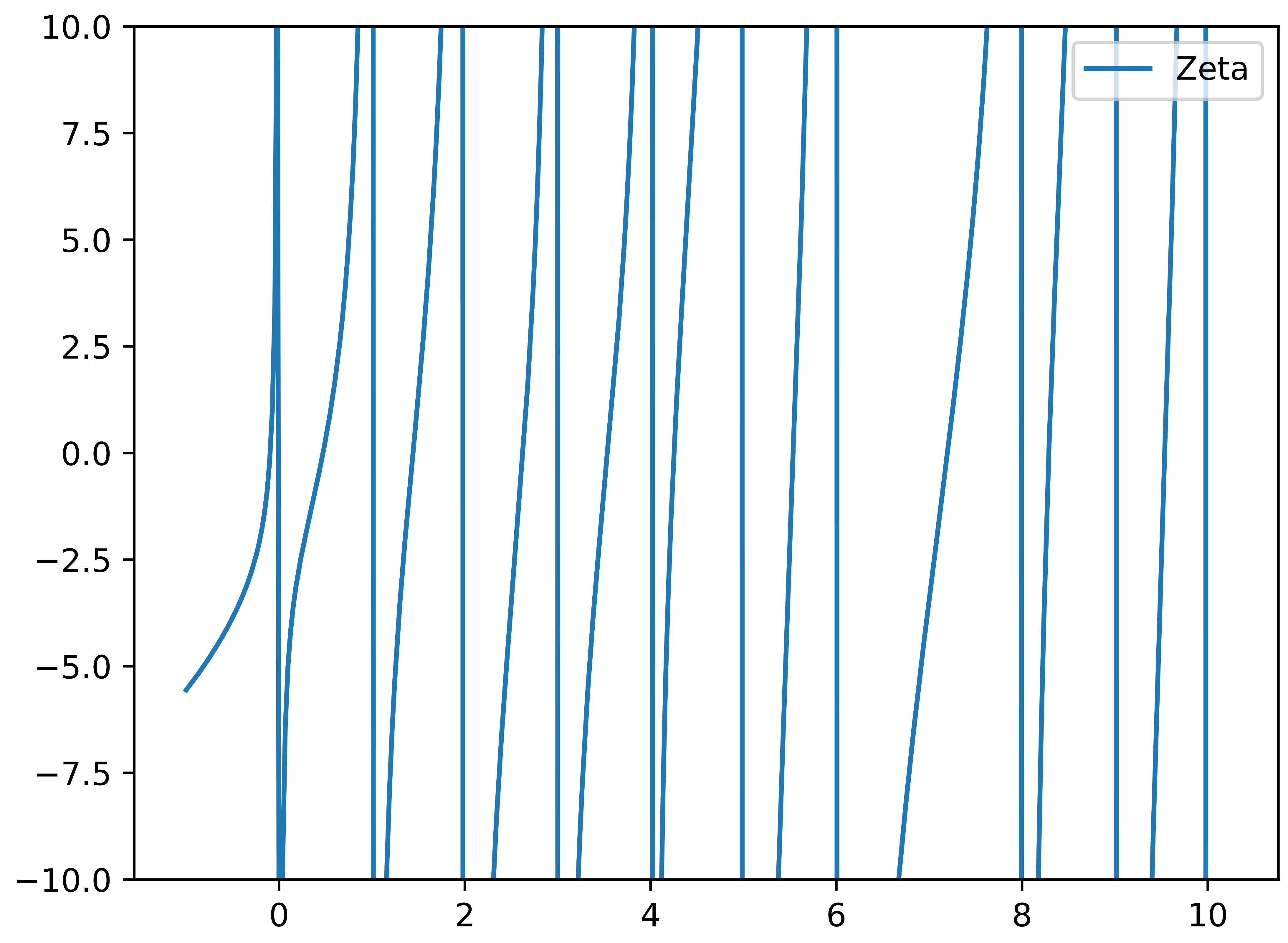


Jenny et al.: Phys. Rev. D 105 (2022)

The Zeta function

$$\mathcal{Z}^{\vec{\mathbf{d}}}_{Jm}(r,q^2) = \sum_{\vec{\mathbf{x}}\in P_{\vec{\mathbf{d}}}} \frac{|\vec{\mathbf{x}}|^J Y_{Jm}(\vec{\mathbf{x}})}{(\vec{\mathbf{x}}^2 - q^2)^r}$$

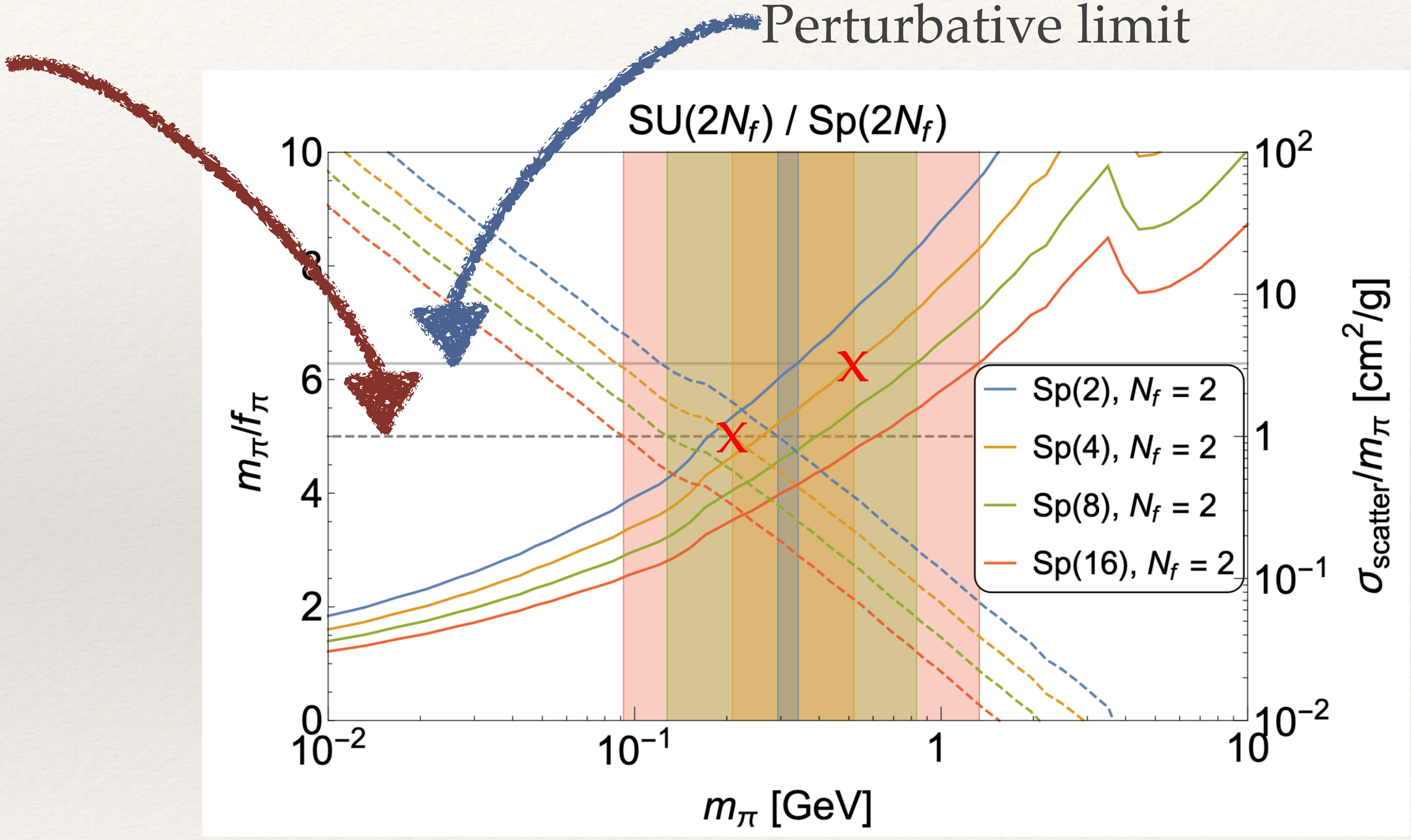
$$P_{\vec{\mathbf{d}}} = \left\{ \vec{\mathbf{x}} \in \mathbb{R}^3 \,\middle|\, \vec{\mathbf{x}} = \vec{\mathbf{y}} + \frac{\vec{\mathbf{d}}}{2}, \vec{\mathbf{y}} \in \mathbb{Z}^3 \right\}$$



Jenny et al.: Phys. Rev. D 105 (2022)

Minimal realisation

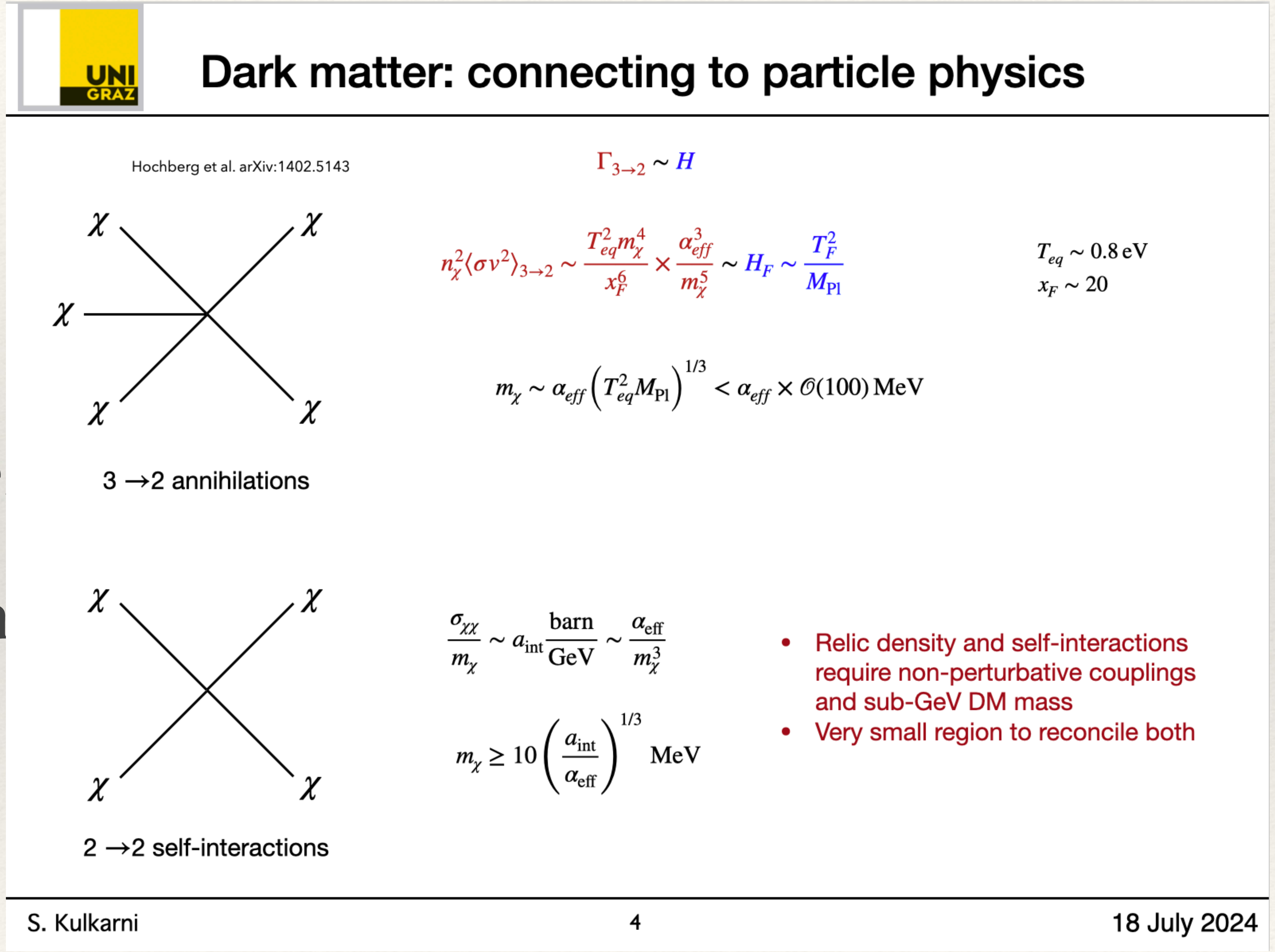
- Parameter space for $Sp(2N_c)$ SIMP models from solving Boltzmann equation
- Why not Sp(2) or Sp(6)?
  - Less constrained
  - Numerically easier
- Large N_C further away from the conformal window for fixed N_f



Hochberg et al. - Phys. Rev. Lett. 115 (2015)

Why confining gauge theory

- Large coupling needed
  - Arises naturally in confining theorie
  - Hard to make it work with elementa



Small-scale structure problems

- Contains ρ, $\pi\pi$, $\pi\pi$
- Trick from before d
- Use young-diagrar

Core-cusp problem: High-resolution simulations show that the mass density profile for CDM halos increases toward the center, scaling approximately as $\rho_{\rm dm} \propto r^{-1}$ in the central region [47, 48, 49]. However, many observed rotation curves of disk galaxies prefer a constant "cored" density profile $\rho_{\rm dm} \propto r^0$ [50, 51, 52], indicated by linearly rising circular velocity in the inner regions. The issue is most prevalent for dwarf and low surface brightness (LSB) galaxies [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65], which, being highly DM-dominated, are appealing environments to test CDM predictions.

Diversity problem: Cosmological structure formation is predicted to be a self-similar process with a remarkably little scatter in density profiles for halos of a given mass [49, 66]. However, disk galaxies with the same maximal circular velocity exhibit a much larger scatter in their interiors [67] and inferred core densities vary by a factor of $\mathcal{O}(10)$ [68].

Missing satellites problem: CDM halos are rich with substructure, since they grow via hierarchical mergers of smaller halos that survive the merger process [69]. Observationally, however, the number of small galaxies in the Local Group are far fewer than the number of predicted subhalos. In the MW, simulations predict $\mathcal{O}(100-1000)$ subhalos large enough to host galaxies, while only 10 dwarf spheroidal galaxies had been discovered when this issue was first raised [70, 71]. Nearby galaxies in the field exhibit a similar underabundance of small galaxies compared to the velocity function inferred through simulations [36, 72, 73].

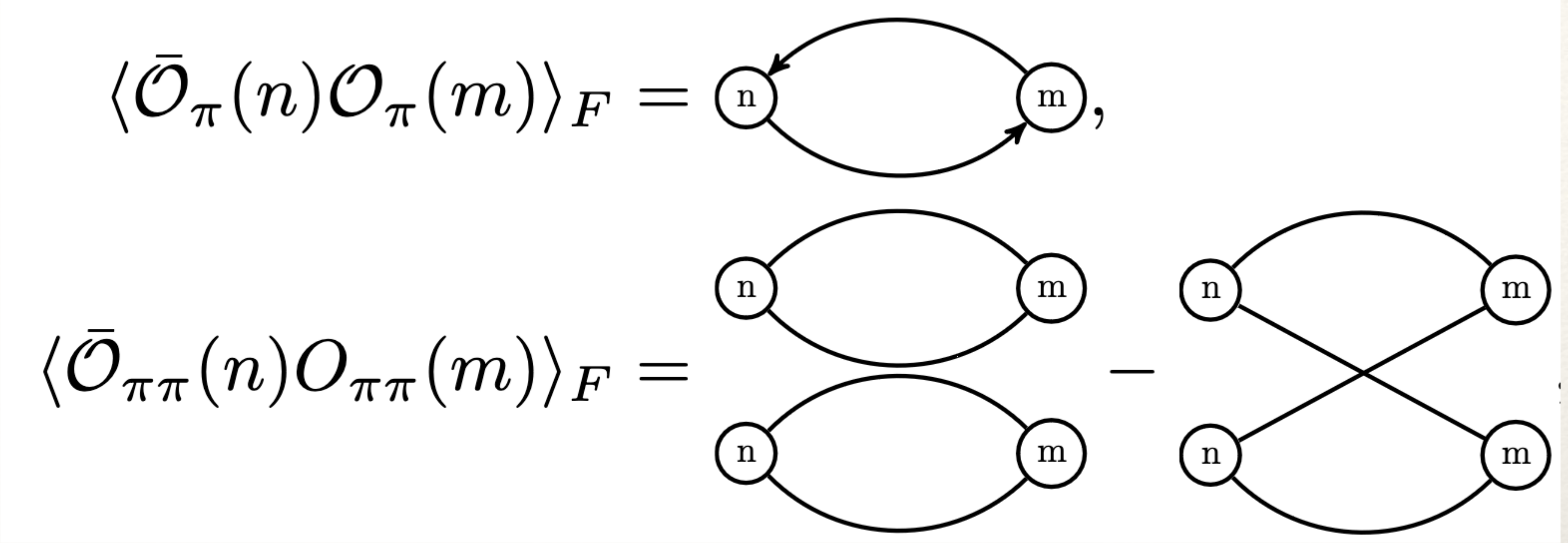
Too-big-to-fail problem (TBTF): In recent years, much attention has been paid to the most luminous satellites in the MW, which are expected to inhabit the most massive suhalos in CDM simulations. However, it has been shown that these subhalos are too dense in the central regions to be consistent with stellar dynamics of the brightest dwarf spheroidals [74, 75]. The origin of the name stems from the expectation that such massive subhalos are too big to fail in forming stars and should host observable galaxies. Studies of dwarf galaxies in Andromeda [76] and the Local Group field [77] have found similar discrepancies.

Sp(4) particle spectrum

Label (M)	Interpolating operator $(\mathcal{O}_M)$	Meson	J^P	$Sp(4)$
PS	$\overline{Q^i}\gamma_5 Q^j$	π	0^-	5
S	$\overline{Q^i} Q^j$	a_0	0^+	5
V	$\overline{Q^i}\gamma_\mu Q^j$	ρ	1^-	10
T	$\overline{Q^i}\gamma_0\gamma_\mu Q^j$	ρ	1^-	10(+5)
AV	$\overline{Q^i}\gamma_5\gamma_\mu Q^j$	a_1	1^+	5
AT	$\overline{Q^i}\gamma_5\gamma_0\gamma_\mu Q^j$	b_1	1^+	10(+5)

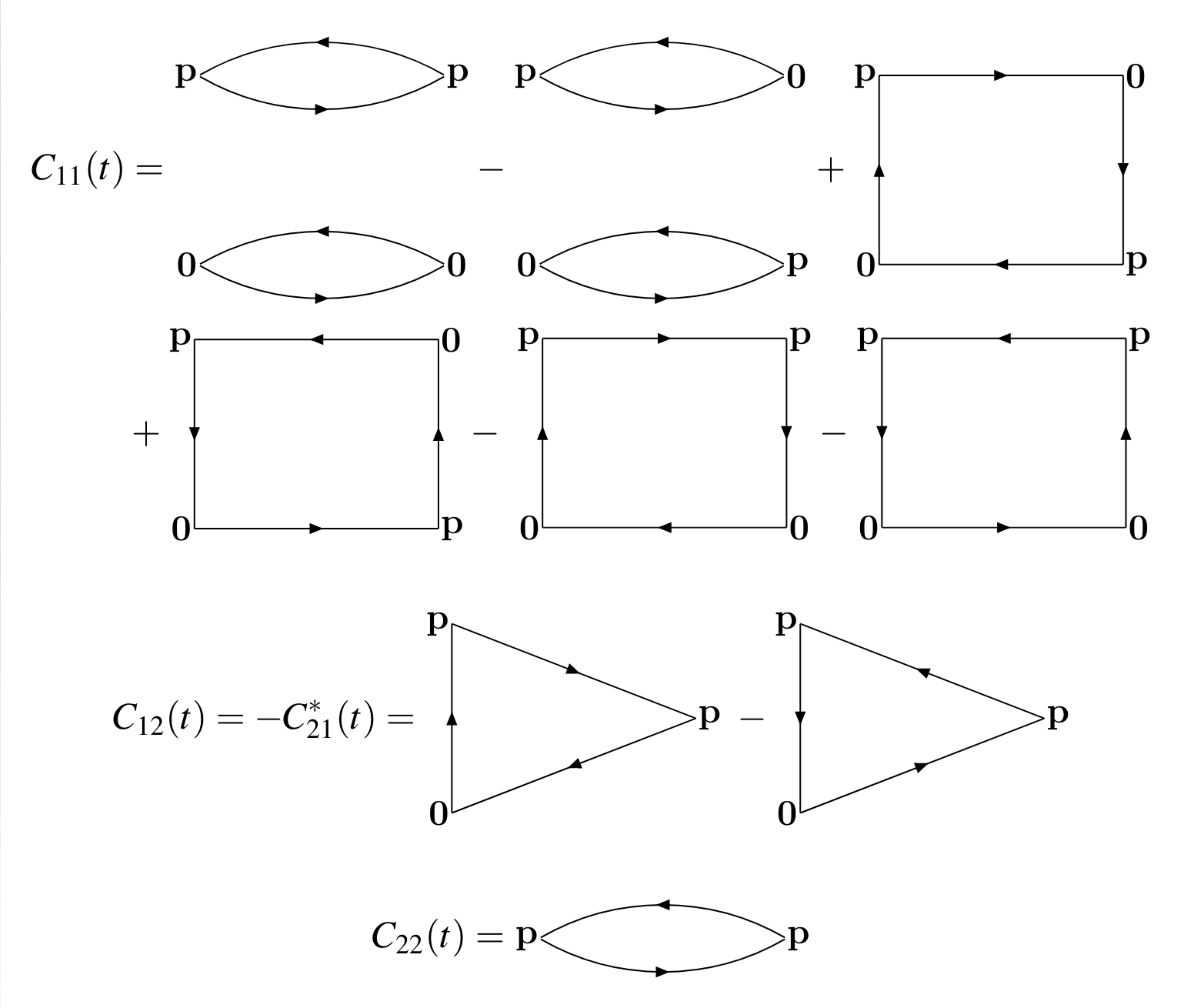
Bennett et al.: arXiv:1909.12662 (2019)

Operators and correlators



- 14-dim: π & $\pi\pi$:

- 10-dim: ρ & $\pi\pi$:



arXiv:1611.09195v1

Energy levels on the lattice

- Each operator in a specified quantum number channel contains the full energy spectrum with some non-trivial (not possible to tell a priori) overlap
  - Solution: Try/use a lot of operators and perform variational analysis

$$C(t) = \left\langle \mathcal{O}(t)\mathcal{O}^{\dagger}(0) \right\rangle = \sum_k \langle 0|\mathcal{O}|k\rangle \left\langle k|\mathcal{O}^{\dagger}|0\right\rangle \exp^{-tE_k}$$

$$\lim_{t\to\infty} C(t) = e^{-tm}$$

- Correlation functions can expressed as diagrams

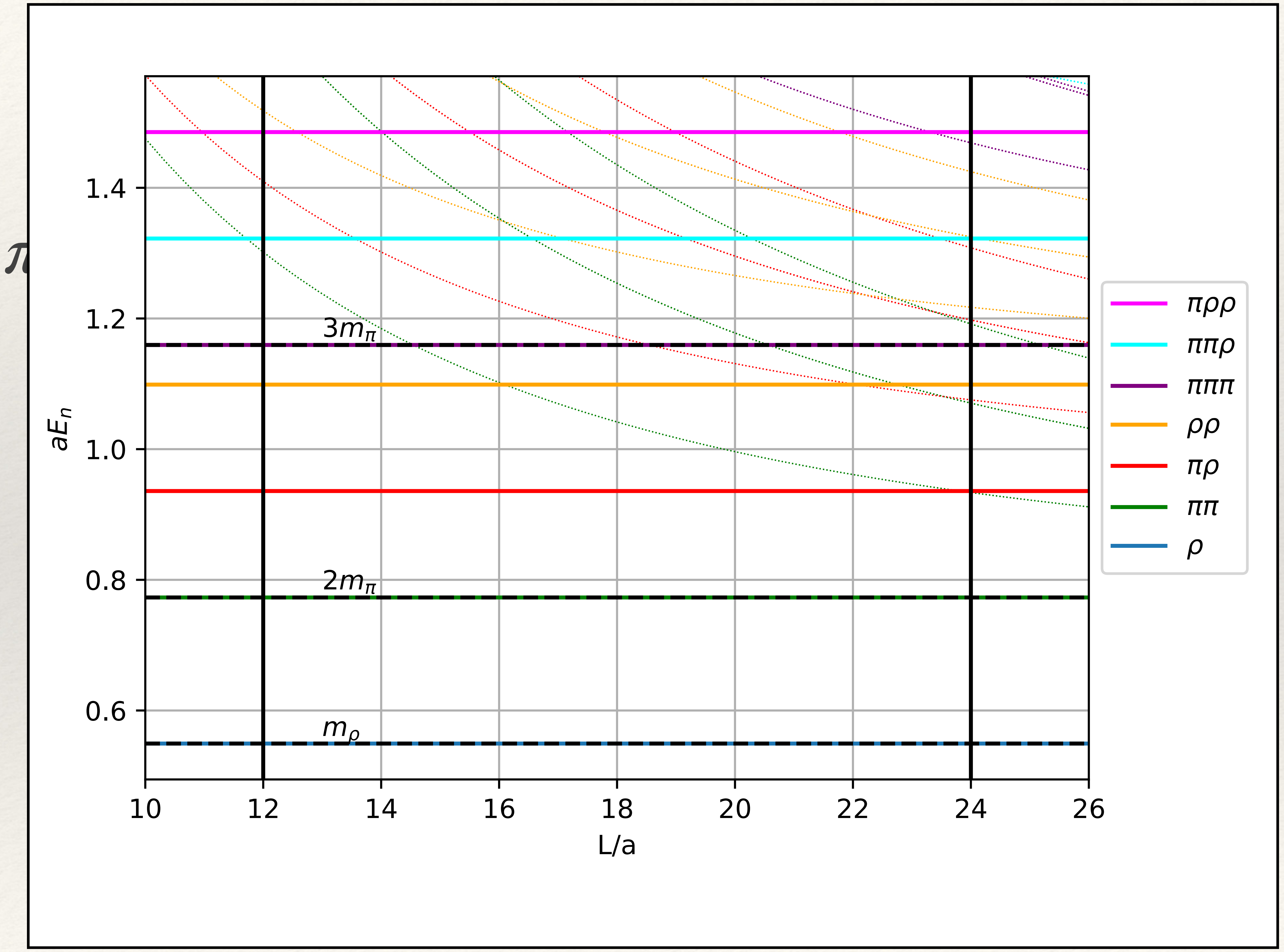
Variational Analysis

$$C_{ij}(t) = \left\langle \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) \right\rangle$$

- Build cross-correlation matrix
- The Eigenvalues of this matrix disentangle the energy levels
- $\lambda_k(t) \propto e^{tE_k}$
- Works best with large operator basis

Trivial energy levels

- Lot of scattering states possible
- Possible operators: ρ, $\pi\pi$, $\pi\rho$, $\rho\rho$, $\pi\pi\pi$, π
- $E = \sum_i \sqrt{m_i^2 + p_i^2}$
- Trivial momenta in finite volume:
  - $p = \frac{2\pi|\vec{n}|}{L}, \vec{n} \in \mathbb{Z}^3$



Lüscher method

Zero momentum $\mathbf{P} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$
(for irrep T_1^- in O_h) [3]:

$$\tan\delta(q) = \frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1; q^2)}$$

Nonzero momentum $\mathbf{P} = (\mathbf{0}, \mathbf{0}, \mathbf{1})\frac{\mathbf{2}\pi}{\mathbf{L}}$
(for irrep A_2^- in D_{4h}) [7]:

$$\tan\delta(q) = \frac{\gamma \pi^{3/2} q^3}{q^2 \mathcal{Z}^{\mathbf{d}}_{00}(1; q^2) + \sqrt{\frac{4}{5}}\, \mathcal{Z}^{\mathbf{d}}_{20}(1; q^2)}$$

P=(1,1,0)
(for irrep B_1^- in D_{2h})

$$\tan\delta(q) = \frac{\gamma \pi^{3/2} q^3}{q^2 \mathcal{Z}^{\mathbf{d}}_{00}(1; q^2) - \sqrt{\frac{1}{5}}\, \mathcal{Z}^{\mathbf{d}}_{20}(1; q^2) + \mathrm{i}\sqrt{\frac{3}{10}}\left(\mathcal{Z}^{\mathbf{d}}_{22}(1; q^2) - \mathcal{Z}^{\mathbf{d}}_{2\bar{2}}(1; q^2)\right)}$$

Lang et. al.: arXiv:1105.5636v3

Phenomenology of scattering channels

- **14**-dim:
  - (Probably) contributes most to $\pi\pi$-scattering
  - 14 out of 25 possible combinations of Pions

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 35$$

$\pi\pi \to \pi\pi$ (I=0,1,2)

$\pi\pi \to \rho$ (I=1)

$\pi\pi \to \pi\pi\pi$ (I=1)

$\pi\pi \to \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Phenomenology of scattering channels

- **1**-dim:
  - (Probably) no large contribution to $\pi\pi$-scattering
  - Mixes in other scattering channel
  - Numerically challenging

$Sp(4)_f$

$5 \otimes 5 = 1 \oplus 10 \oplus 14$

$10 \otimes 5 = 5 \oplus 10 \oplus 35$

$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 35$

$\pi\pi \rightarrow \pi\pi$ (I=0,1,2)

$\pi\pi \rightarrow \rho$ (I=1)

$\pi\pi \rightarrow \pi\pi\pi$ (I=1)

$\pi\pi \rightarrow \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Phenomenology of scattering channels

- **10**-dim:
  - Mixing with the Rho
  - $\pi\pi\pi \rightarrow \pi\pi$
- Work in progress

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 35$$

$\pi\pi \rightarrow \pi\pi$ (I=0,1,2)

$\pi\pi \rightarrow \rho$ (I=1)

$\pi\pi \rightarrow \pi\pi\pi$ (I=1)

$\pi\pi \rightarrow \pi\pi\rho$ (I=0,1,2)

etc.

Feger et al.: Comp.Phys.Com 257 (2020)

Phenomenology of scattering channels

- **14**-dim:
  - Makes up most $\pi\pi$ scattering (14/25)
  - Easiest on the lattice
- **10**-dim:
  - Mixing with dark ρ
  - $\pi\pi\pi \to \pi\pi$
- **1**-dim:
  - Mixing with other states

$$Sp(4)_f$$

$$5 \otimes 5 = 1 \oplus 10 \oplus 14$$

$$10 \otimes 5 = 5 \oplus 10 \oplus 35$$

$$5 \otimes 5 \otimes 5 = 3(5) \oplus 10 \oplus 30 \oplus 35$$

$\pi\pi \to \pi\pi$ (I=0,1,2)

$\pi\pi \to \rho$ (I=1)

$\pi\pi \to \pi\pi\pi$ (I=1)

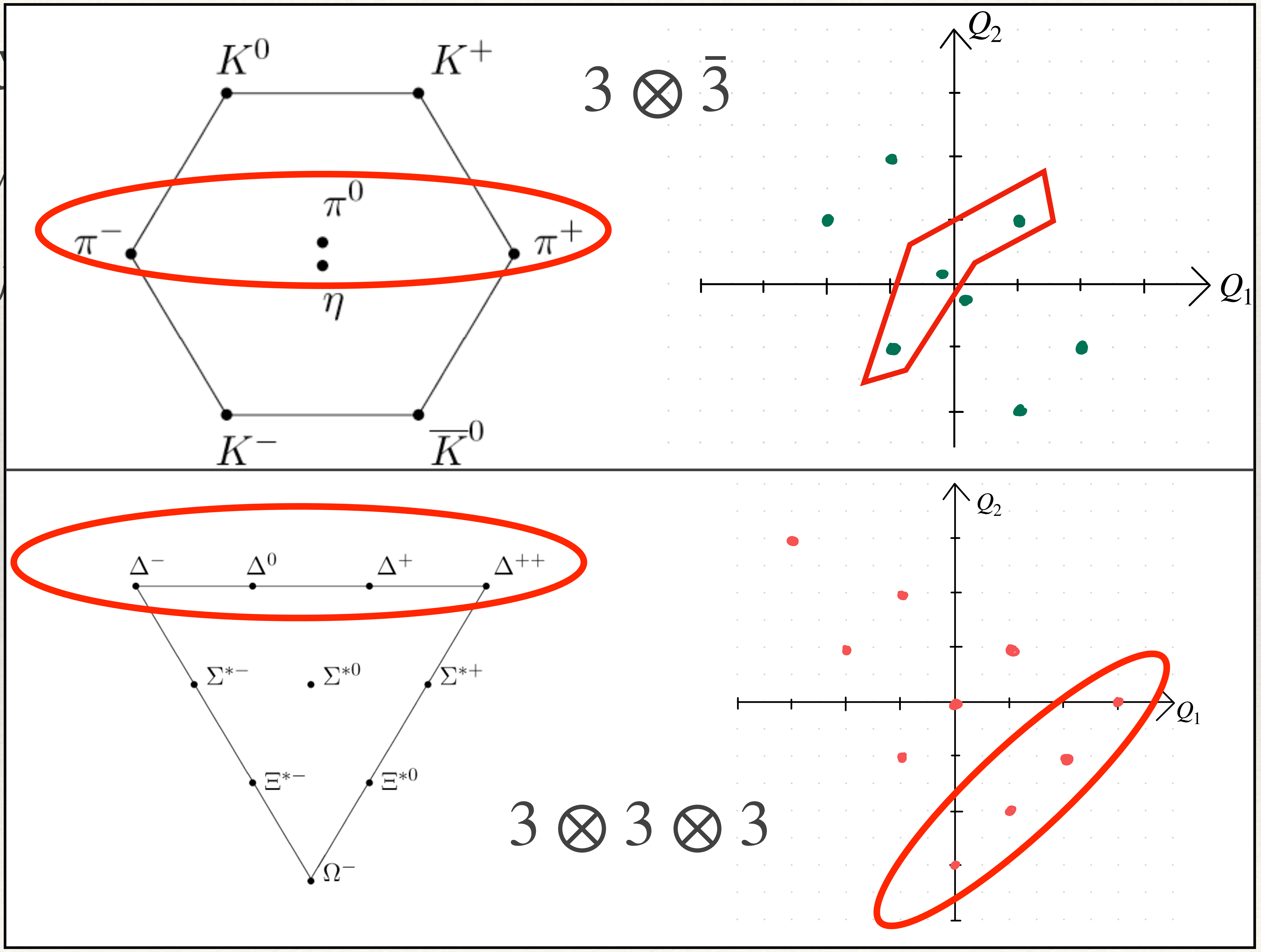
$\pi\pi \to \pi\pi\rho$ (I=0,1,2)

etc.

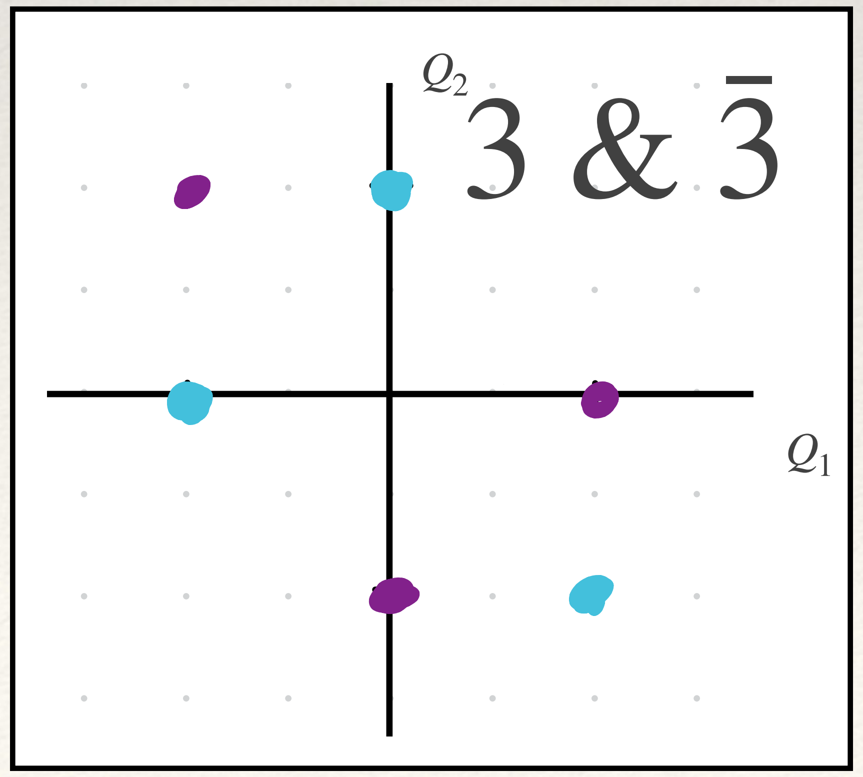
Feger et al.: Comp.Phys.Com 257 (2020)

Flavour quantum numbers

- Composite states live in irreps of the flavour sy
- Can be represented in diagrams given by the w
  - "Meson-octet" and "Baryon-Decuplet" in *SU*
  - Mass-degenerate → perfect symmetry

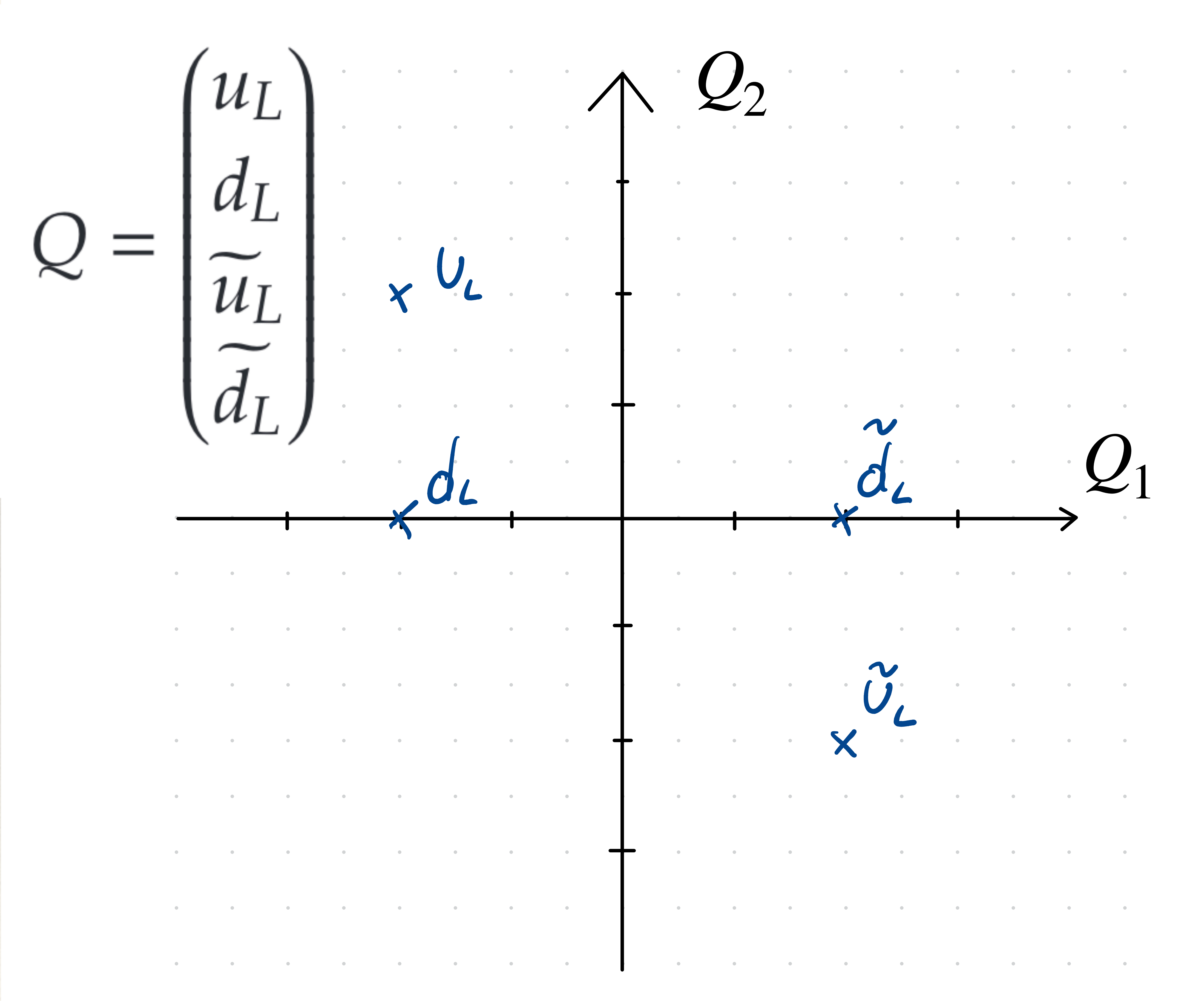


Weight system of the fundamental of SU(3):



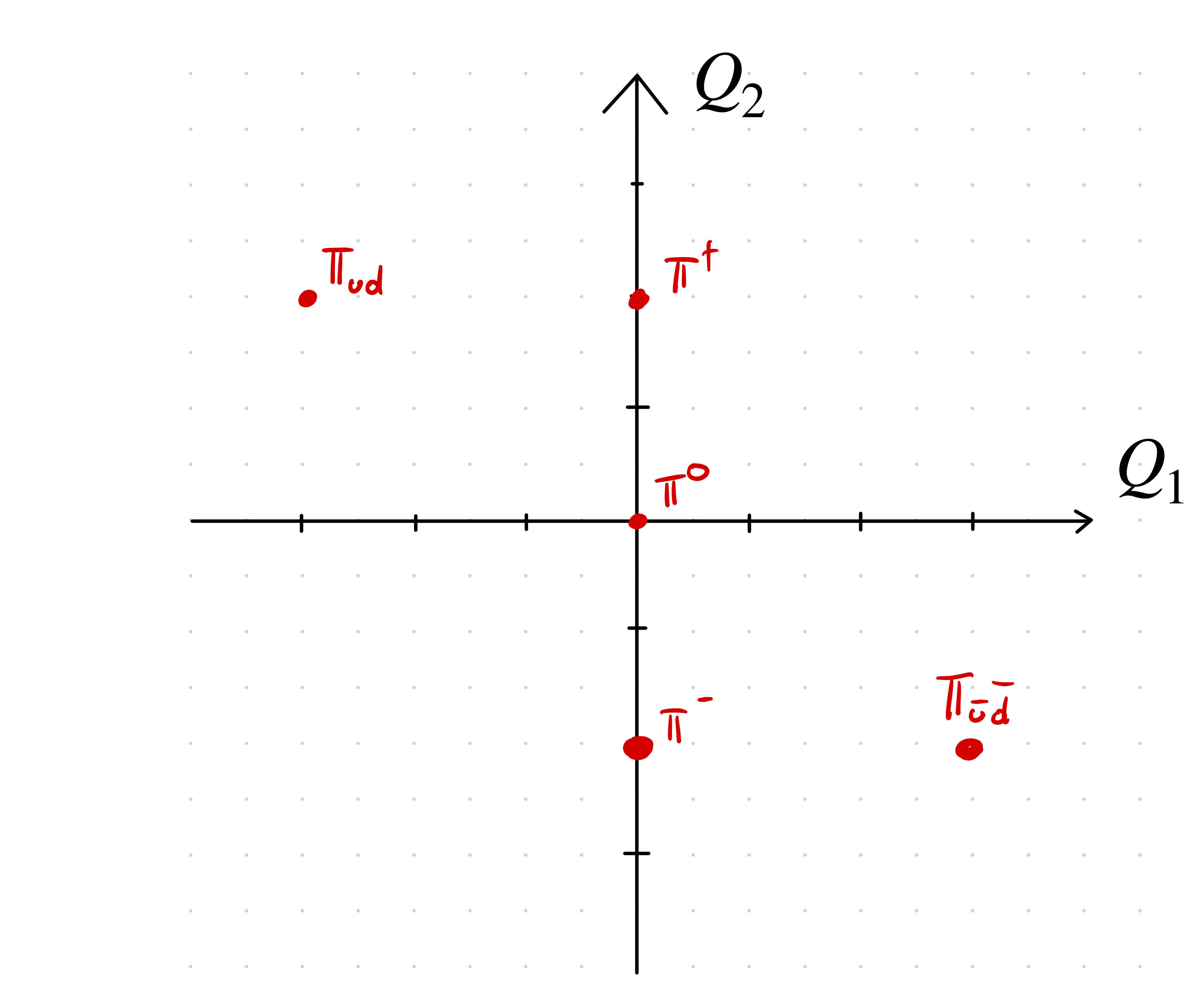
Flavour quantum numbers in Sp(4)

- Similar in Sp(4) for visualising scatter
- Quarks in fundamental of Sp(4) (**4**-ple
- $4 \otimes 4 = 1 \oplus 5 \oplus 10$
  - Pions in **5**
  - Rhos in **10**



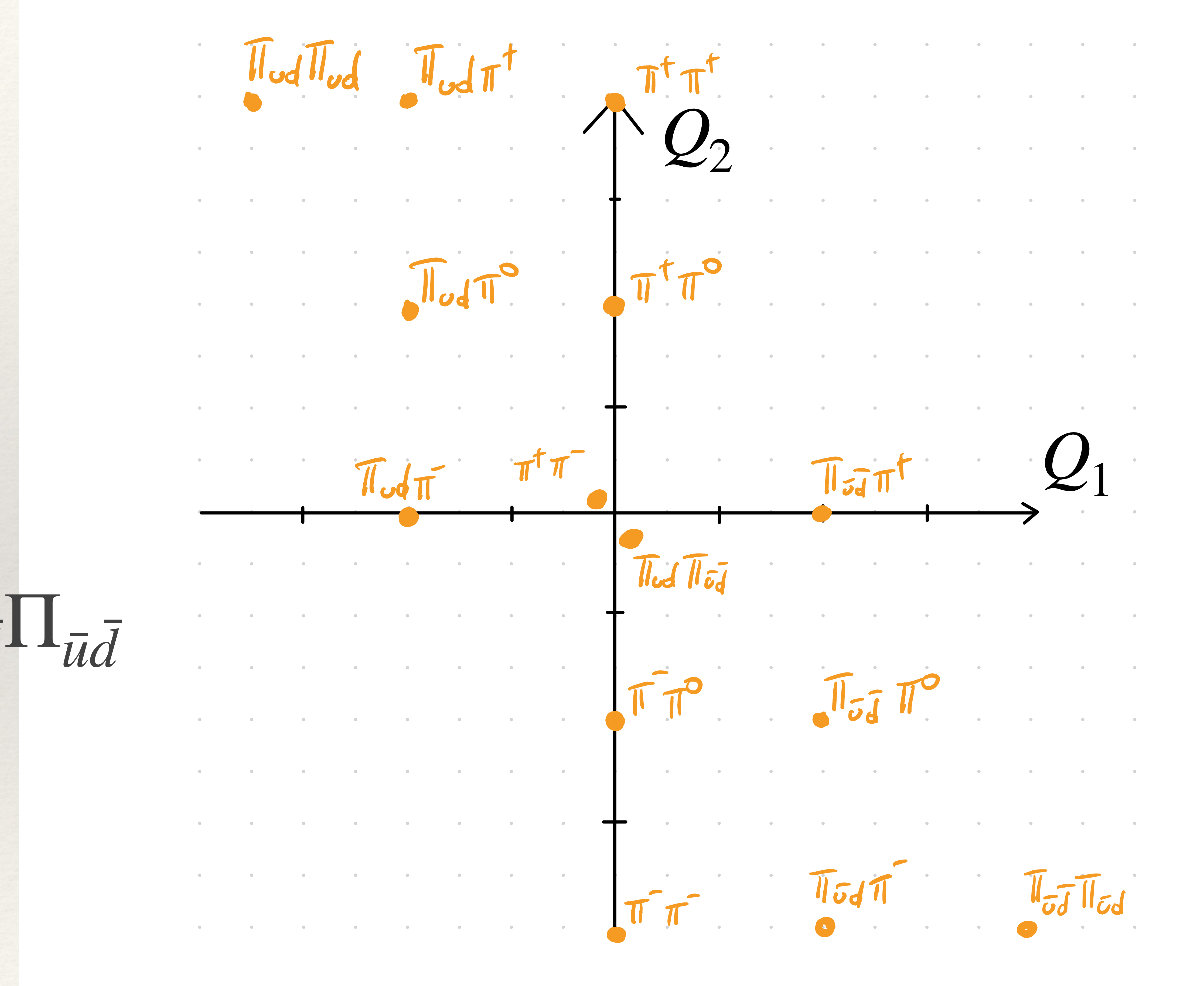
Pions form a **5**-plet

- Isomorphism: SO(5) = Sp(4)
- Quark content can be read off
  - $\pi^+ = u\gamma_5\bar{d}$
- Scattering states:
  - $5 \otimes 5 = 1 \oplus 10 \oplus 14$



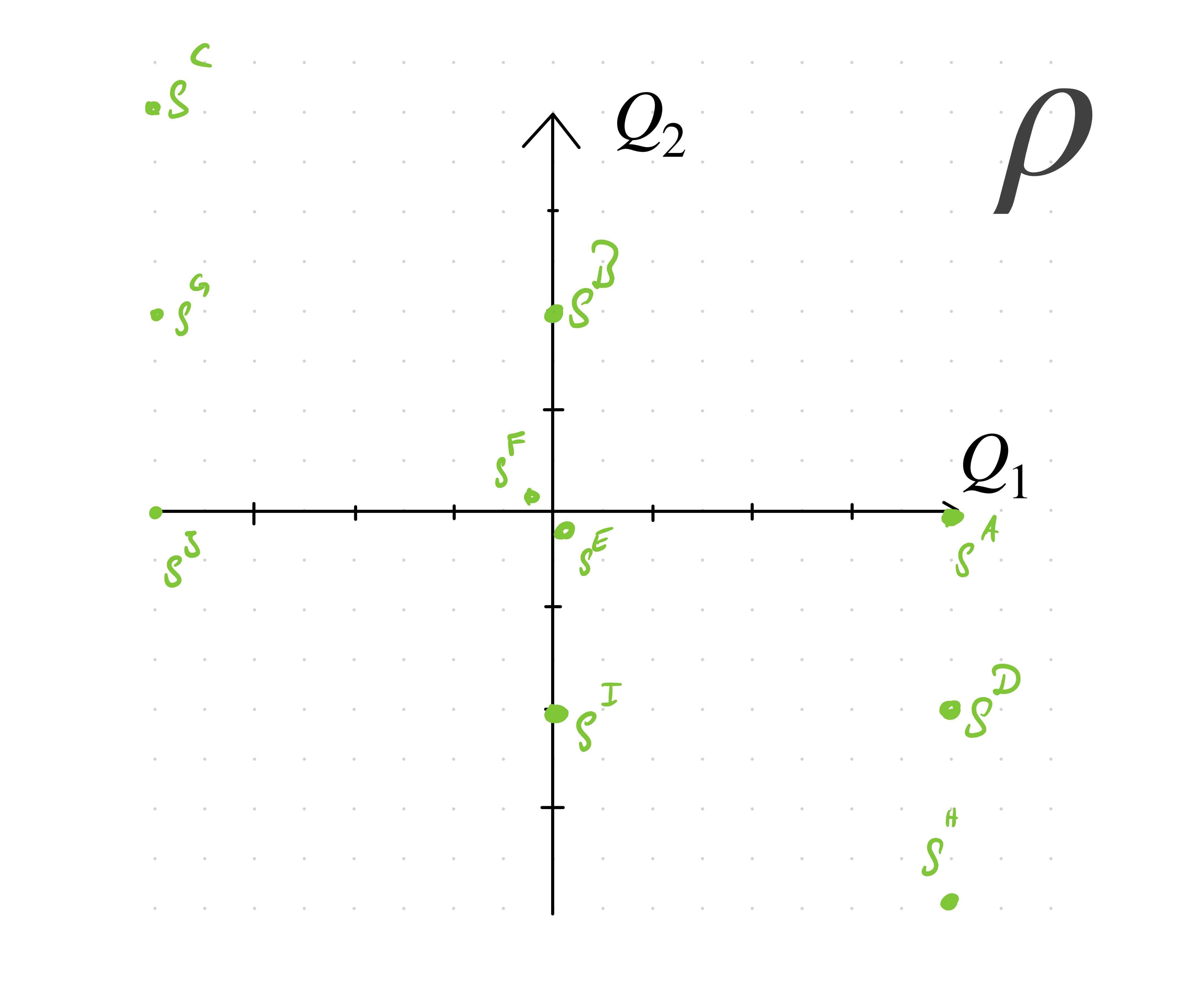
The **14**-plet

- Reminder: $5 \otimes 5 = 1 \oplus 10 \oplus 14$
- $\pi^+\pi^+$ is unique to the **14**
- $\mathcal{O}^{14}_{\pi\pi} = \pi^+\pi^+ = \pi^-\pi^- = \Pi_{ud}\Pi_{ud} = \Pi_{\bar{u}\bar{d}}\Pi_{\bar{u}\bar{d}}$



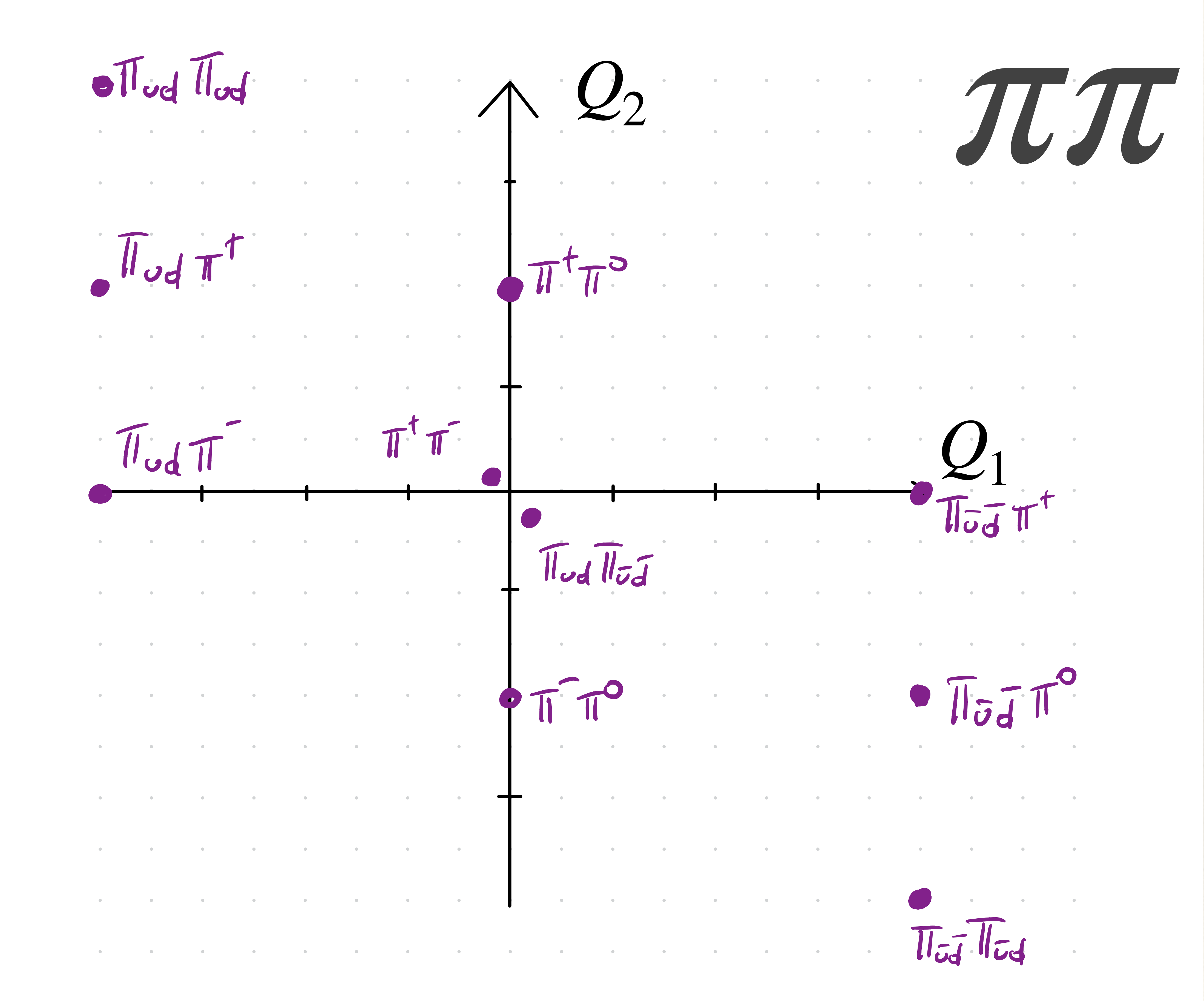
The **10**-plet

- Contains ρ, $\pi\pi$, $\pi\pi\pi$
- Trick from before does not work
- Use young-diagrams



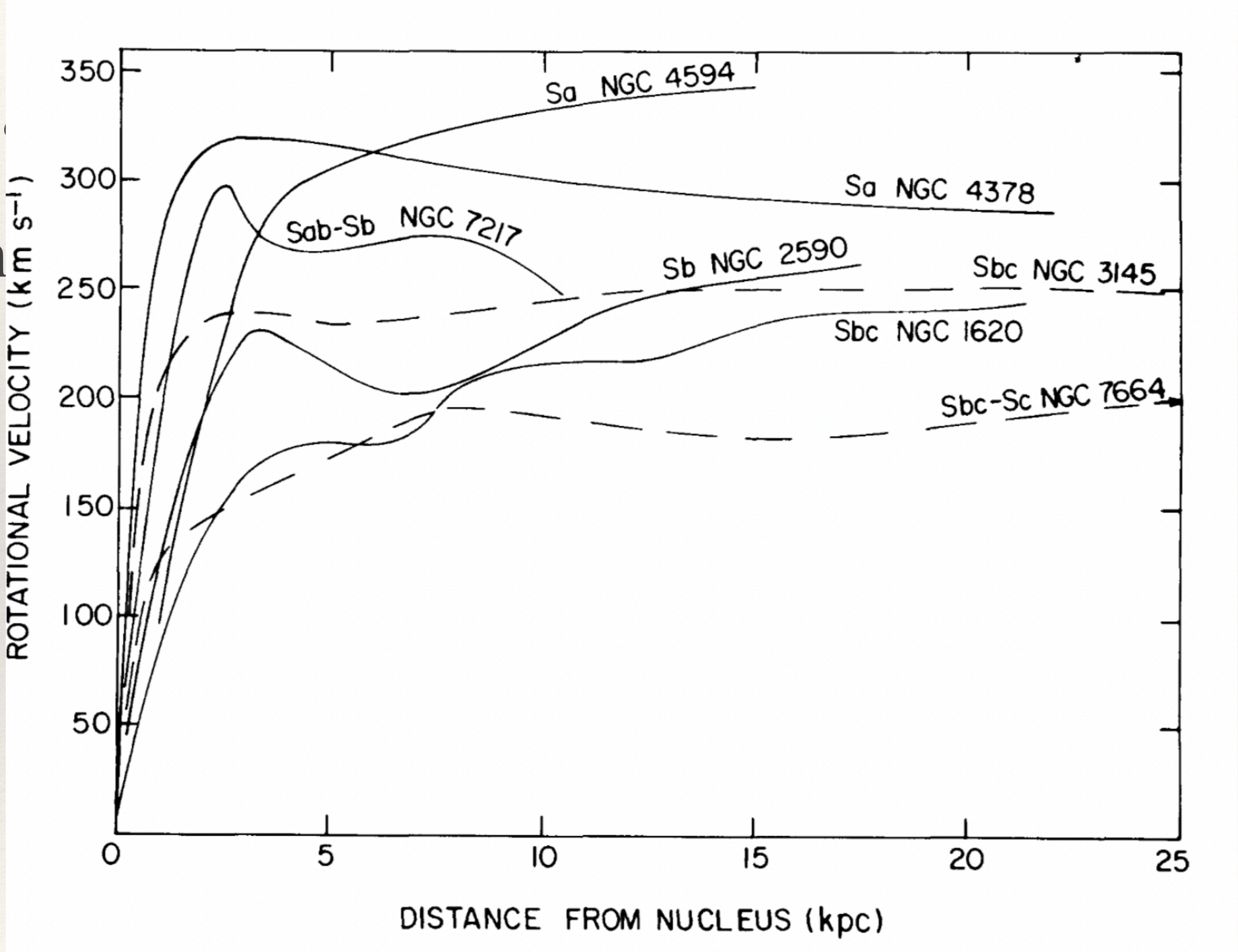
The **10**-plet

- Contains ρ, $\pi\pi$, $\pi\pi\pi$
- Trick from before does not work
- Use young-diagrams



Dark Matter

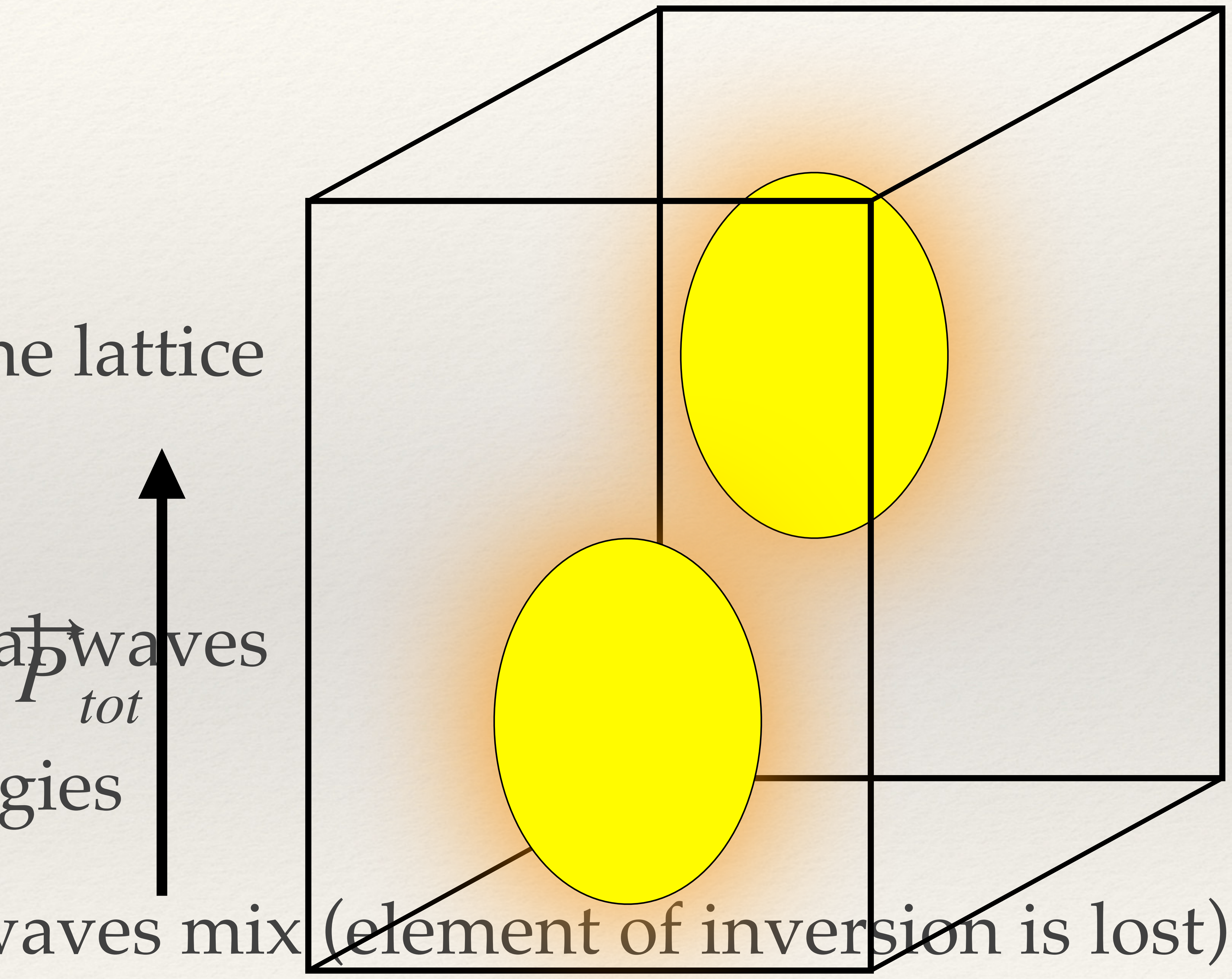
- Collection of phenomena with no expl
  - Rotation curves, structure formation
- Possible explanations:
  - Modified gravity
  - Non observable form of matter
  - Particle beyond the SM



Rubin et al.: Ap.J.L. 225 (1978)

Next steps: $\pi\pi \to \rho$

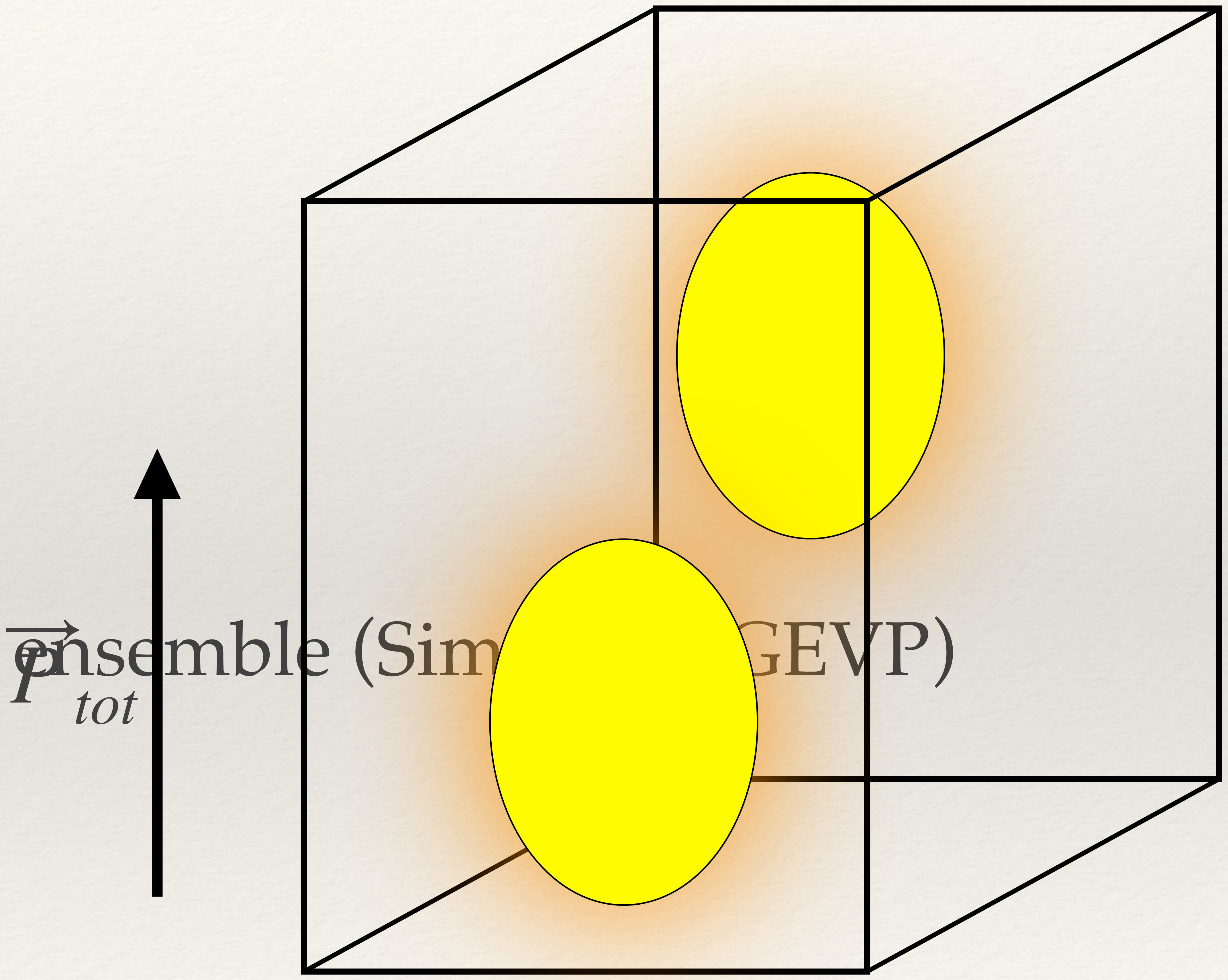
- Important for phenomenology
- Quantization condition is more involved
  - Finite momenta change symmetries on the lattice
  - From O_h to little groups (D_{4h}, D_{2h}, …)
- Different lattice irreps probe different partial waves
- Lowest partial wave dominates @ low energies
- For unequal masses even and odd partial waves mix (element of inversion is lost)



YD, F. Zierler and A. Maas: arXiv:2405.06506

What to do?

- Project operators in the desired irrep
- Done by relatively simple formula
- Wick contractions do not change
- Access to a lot more datapoints from one ensemble (Sim ... GEVP)
- Formulas for $\delta(E, L)$ change



YD, F. Zierler and A. Maas: arXiv:2405.06506

Trivial energy levels

- Lot of scattering states possible
- Possible operators: ρ, $\pi\pi$, $\pi\rho$, $\rho\rho$, $\pi\pi\pi$, π...
- $E = \sum_i \sqrt{m_i^2 + p_i^2}$
- Trivial momenta in finite volume:
  - $p = \frac{2\pi|\vec{n}|}{L}, \vec{n} \in \mathbb{Z}^3$