Jožef Stefan Institute
Ljubljana, Slovenia

UNIVERSITY OF LJUBLJANA
FMF
Faculty of Mathematics and Physics

Nikiša Plešec

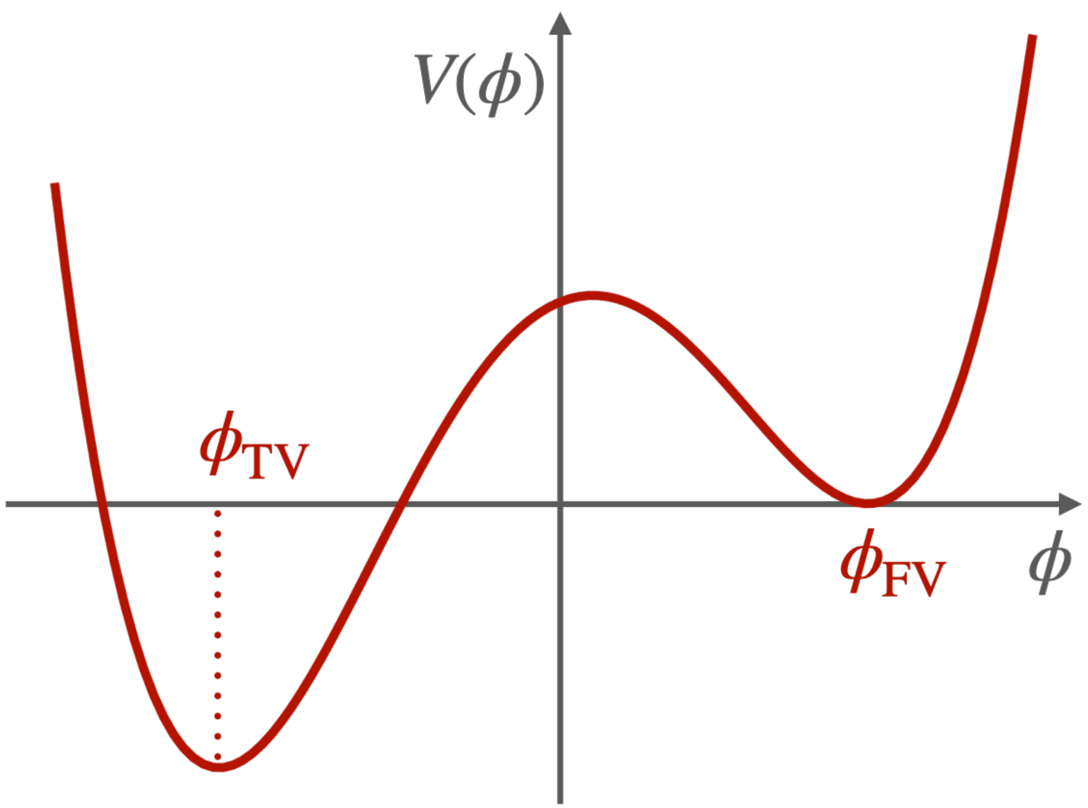
False Vacuum Decay Rate of a Scalar Field at One Loop

Belica, October 4, 2024

Introduction

- **False vacuum decay**: tunnelling from false vacuum (FV) to true vacuum (TV)

- **Motivation**:
  - $D = 4$: metastable Standard Model (SM) potential
  - $D = 3$: early universe
  - $D = 2$: spin chains

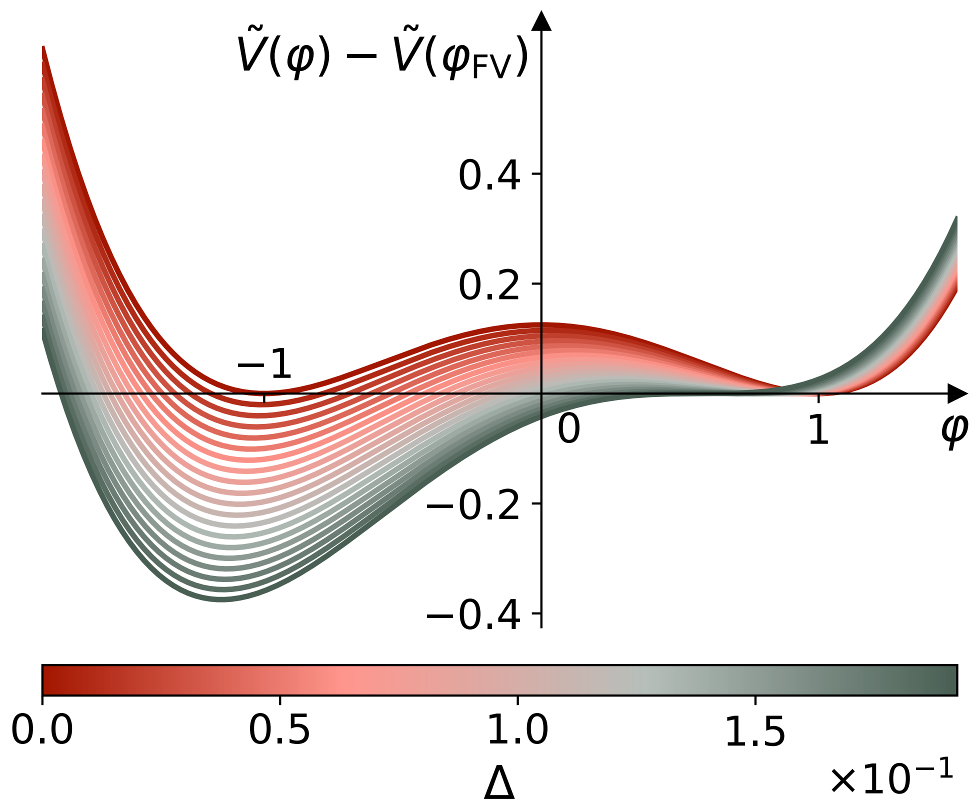


- General renormalizable potential of a real scalar field ϕ:

$$V(\phi) = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 \left(\phi - v\right) = \lambda v^2 \tilde{V}(\varphi) \qquad \varphi = \phi / v$$

Problem Framework

- $0 < \lambda \ll 1$ and $0 < \Delta \ll 1$
- Dimensionless coupling Δ, $\Delta_{\text{max}} = 1/\sqrt{27} \approx 0.19245$



- Decay rate in general dimensions

$$\frac{\Gamma}{V} = \left(\frac{S_R}{2\pi\hbar}\right)^{D/2} \left| \frac{\det'\left[-\partial^2 + m_{\text{B}}^2\right]}{\det\left[-\partial^2 + m_{\text{FV}}^2\right]} \right|^{-1/2} e^{-S_R/\hbar - S_{\text{ct}}}$$

$$m_{\text{B}}^2 = V''(\bar{\phi}), \quad m_{\text{FV}}^2 = V''(\phi_{\text{FV}})$$

- $\det'$: without zero eigenvalues
- The decay rate should vanish at $\Delta = 0$ and $\Delta = \Delta_{\text{max}}$

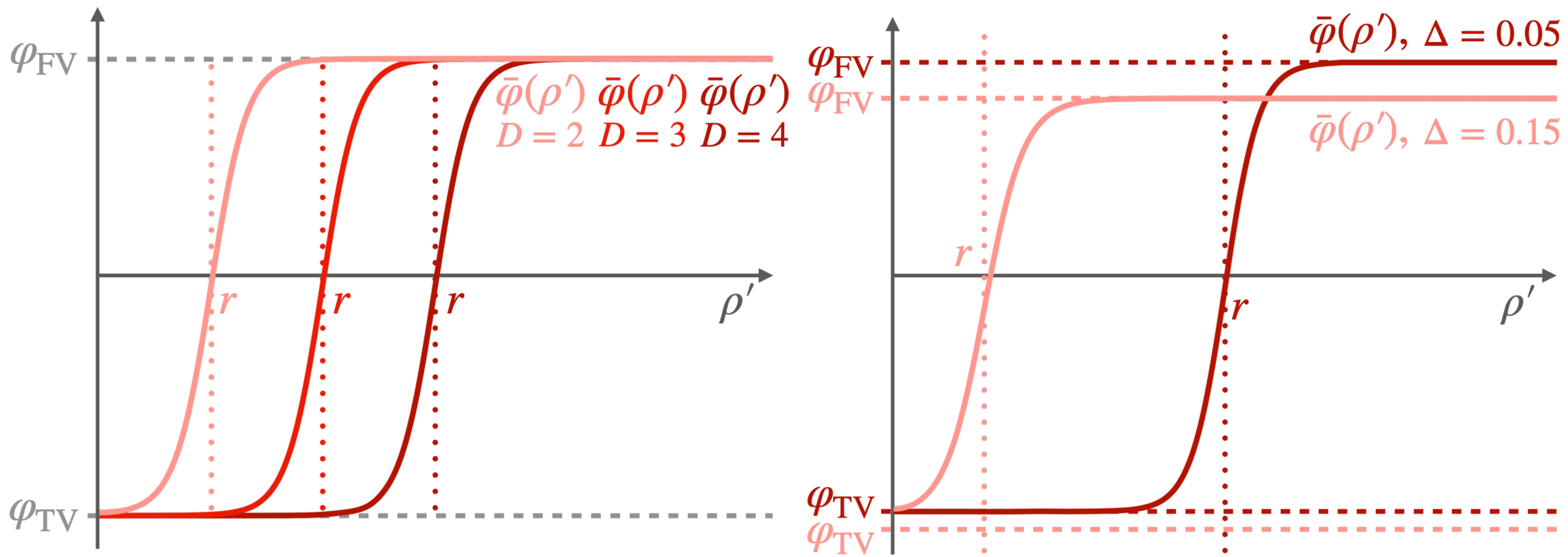
Bounce Solution

- Unstable instanton configuration
- Extremizes the (Euclidean) action
- Radial $O(D)$ symmetry

$$\varphi'' + \frac{D-1}{\rho'}\varphi' - \frac{\partial \tilde{V}}{\partial \varphi} = 0$$

$$\rho' = \sqrt{\lambda v^2}\rho$$

$$\bar{\varphi}'(0) = \bar{\varphi}'(\infty) = 0, \quad \bar{\varphi}(0) = \varphi_{\text{in}}, \quad \bar{\varphi}(\infty) = \varphi_{\text{FV}}$$



Bounce Solution

- Thin-wall limit:

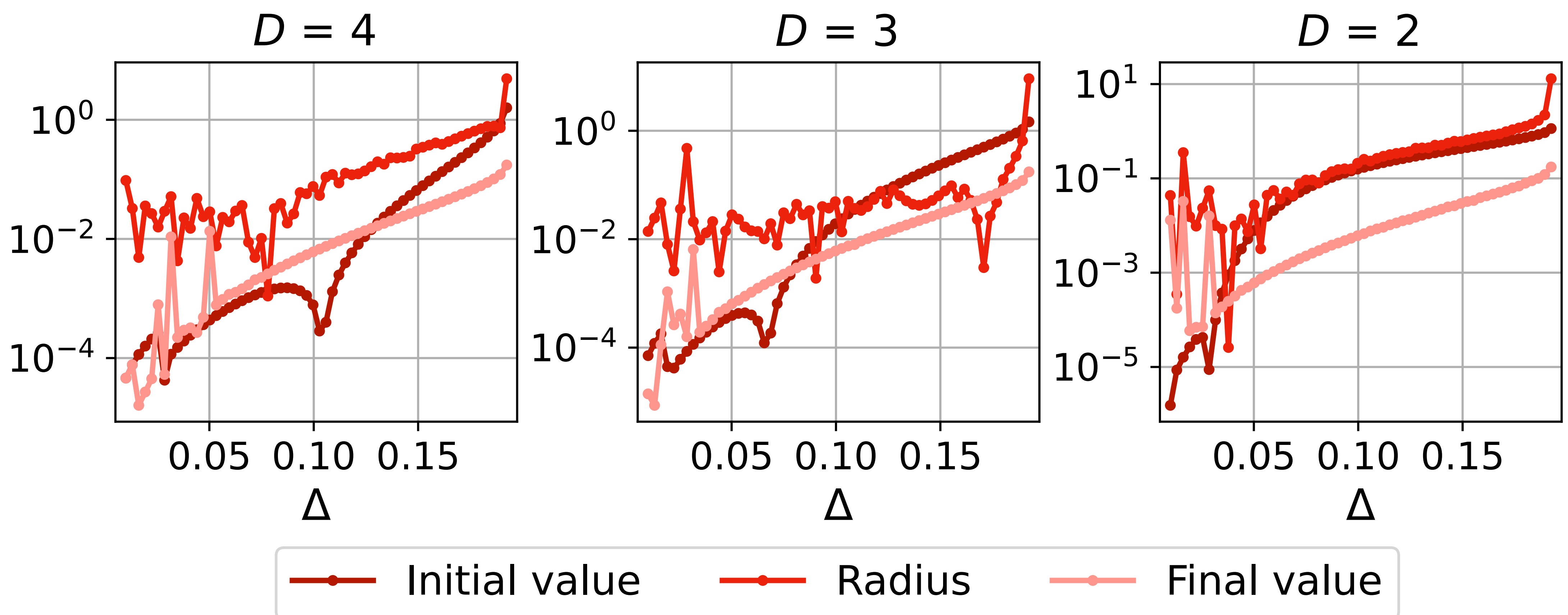
$$\bar{\varphi} = \sum_{n=0}^{\infty} \Delta^n \varphi_n \quad r = \frac{1}{\Delta} \sum_{n=0}^{\infty} \Delta^n r_n \longrightarrow$$

first order $\mathcal{O}(\Delta^0)$

$$\varphi_0(z) = \tanh(z/2)$$
$$r_0 = (D-1)/3$$

$z = \rho' - r$

- Absolute deviation of the thin-wall solution of order $\mathcal{O}(\Delta^2)$:



- $\varphi_{\text{in}} \neq \varphi_{\text{TV}}$ for numerical solution

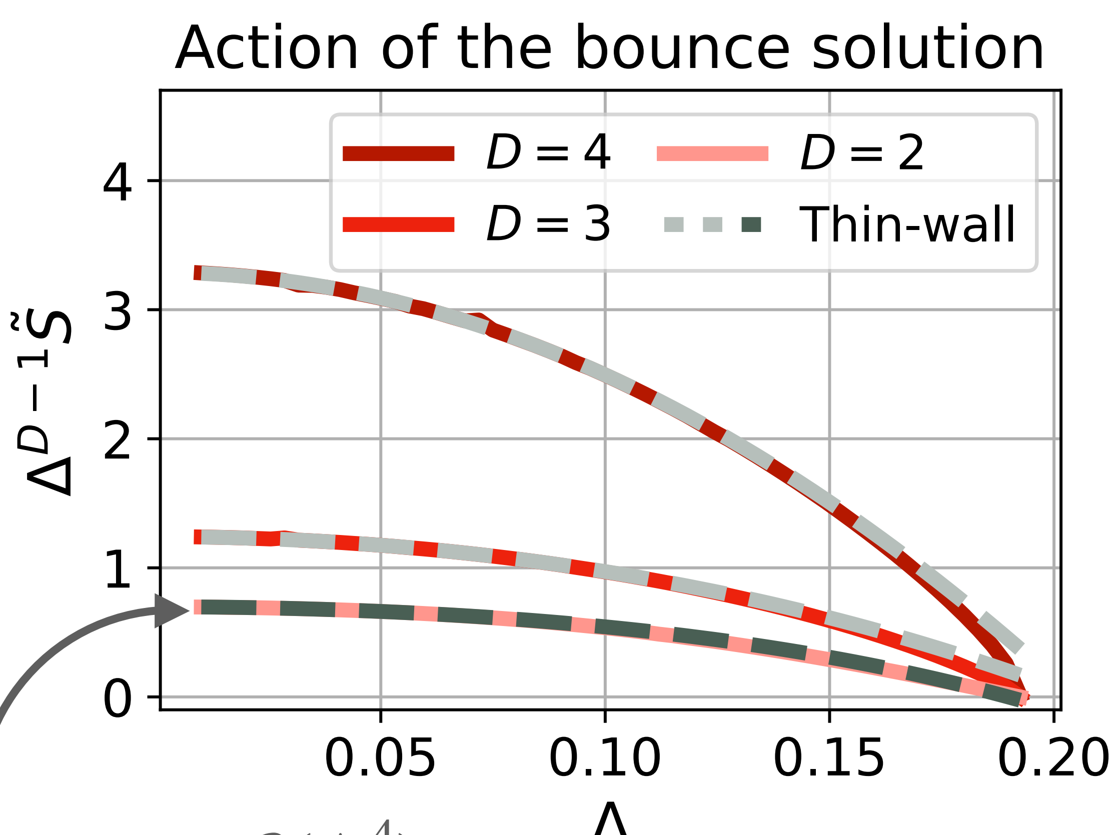
Bounce Action

$$S = \frac{v^{4-D}}{\lambda^{D/2-1}} \frac{\Omega_D}{\Delta^{D-1}} \int_0^\infty \mathrm{d}\rho' \left(\Delta\rho'\right)^{D-1} \left[\frac{1}{2}\varphi'^2 + \tilde{V}(\varphi) - \tilde{V}(\varphi_{\mathrm{FV}})\right] = \frac{v^{4-D}}{\lambda^{D/2-1}} \frac{\Omega_D}{\Delta^{D-1}} \tilde{S}$$

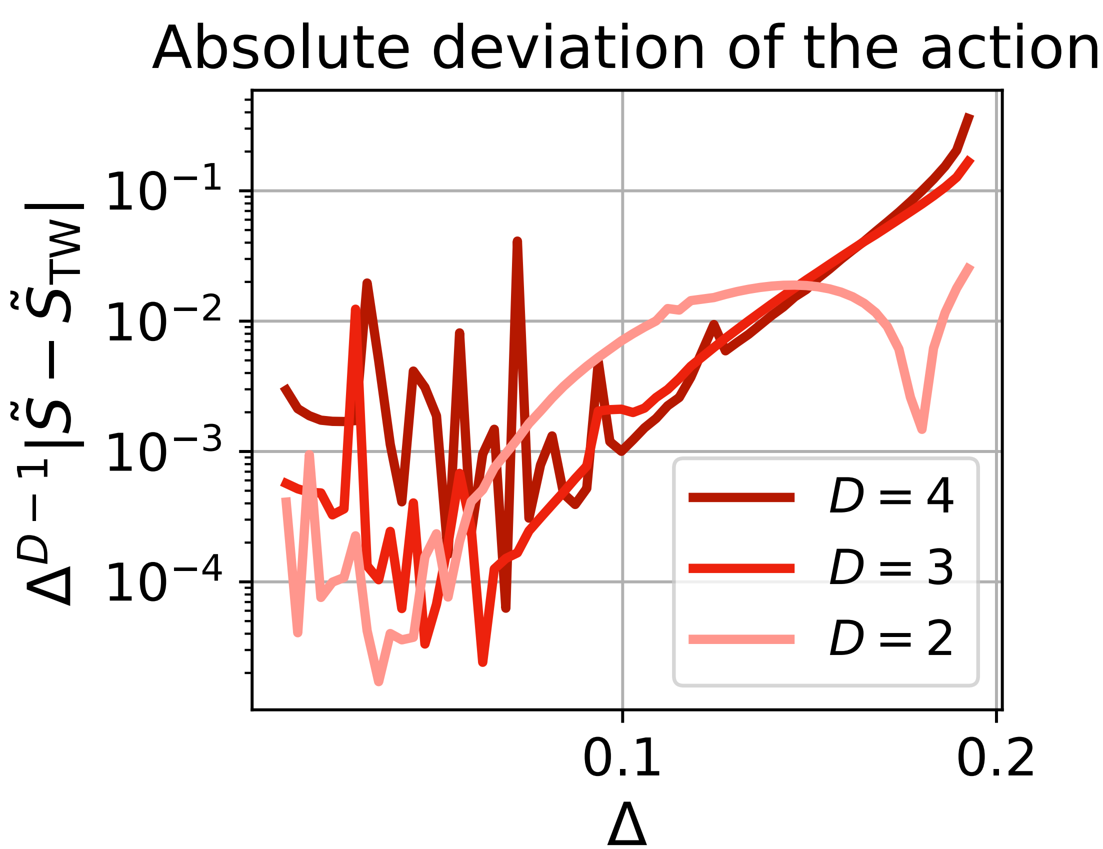
$\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$

first order $\mathcal{O}(\Delta^0)$

- Thin wall limit: $\tilde{S} = \sum_{n=0}^{\infty} \Delta^n \tilde{S}_n \longrightarrow \tilde{S}_0 = \frac{\Omega_D}{\Delta^{D-1}} \frac{2}{3D} \left(\frac{D-1}{3}\right)^{D-1}$



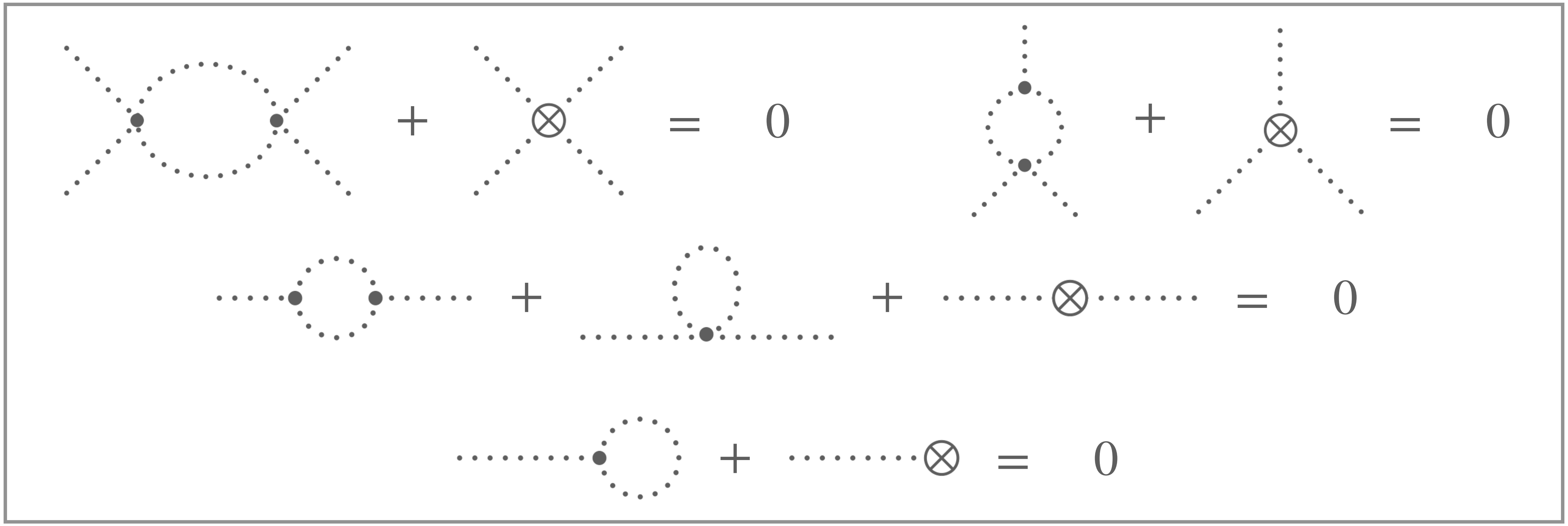
of order $\mathcal{O}(\Delta^4)$



Renormalized Bounce Action

- Counterterm potential: $$V_{\rm ct}(\phi) = \frac{\delta_\lambda}{8}\phi^4 - \frac{\lambda\delta_v}{4}\phi^2 + \lambda v^3 \delta_\Delta \phi$$

- Counterterms cancel divergencies at one loop:



- Dimensional regularization with minimal subtraction (MS) scheme

Renormalized Bounce Action

- $D = 4$

Counterterms

$$\delta_\Delta = 0, \quad \delta_\lambda = \frac{9\lambda^2}{(4\pi)^2\varepsilon}, \quad \delta_v = \frac{3\lambda v^2}{(4\pi)^2\varepsilon}$$

Renormalization group running

$S \propto \lambda^{-1}$

$$\lambda(\mu) = \lambda_0 + \frac{9\lambda_0^2}{(4\pi)^2} \ln \frac{\mu}{\mu_0}$$

$$S_R + S_{\rm ct} = S \left[1 - \frac{9\lambda_0}{(4\pi)^2} \left(\frac{1}{\varepsilon} + \ln \frac{\mu}{\mu_0} \right) \right]$$

- $D = 2$

Counterterms

$$\delta_\Delta = 0, \quad \delta_\lambda = 0, \quad \delta_v = \frac{3}{2\pi\varepsilon}$$

Renormalization group running

$S \propto v^2$

$$v^2(\mu) = v_0^2 + \frac{3}{2\pi} \ln \frac{\mu}{\mu_0}$$

$$S_R + S_{\rm ct} = S \left[1 + \frac{15}{4\pi v_0^2} \left(\frac{1}{\varepsilon} + \ln \frac{\mu}{\mu_0} \right) \right]$$

- In $D = 3$ all counterterms vanish in MS scheme

Quantum Fluctuations

- Contribution from quantum fluctuations:

$O(D)$ symmetry

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-1/2} = \left|\frac{\det'\left[-\partial^2 + m_{\rm B}^2\right]}{\det\left[-\partial^2 + m_{\rm FV}^2\right]}\right|^{-1/2} = \left|\prod_{\ell=0}^{\infty}\frac{\det' \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}}\right|^{-1/2}$$

- Gel'fand Yaglom:

$$\mathcal{O}_\ell\,\psi_\ell(\rho) = 0, \quad \mathcal{O}_{{\rm FV},\ell}\,\psi_{{\rm FV},\ell}(\rho) = 0 \qquad \psi_{({\rm FV}),\ell}\,(\rho \sim 0) = \rho^\ell$$

$$\frac{\det \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}} = \lim_{\rho\to\infty}\left(\frac{\psi_\ell(\rho)}{\psi_{{\rm FV},\ell}(\rho)}\right)^{d_\ell} = R_\ell(\rho \to \infty)^{d_\ell}$$

$$\psi_{{\rm FV},\ell}(\rho \sim 0) = \rho^\ell, \quad R_\ell(\rho = 0) = 1, \quad R'_\ell(\rho = 0) = 0$$

degeneracy factor

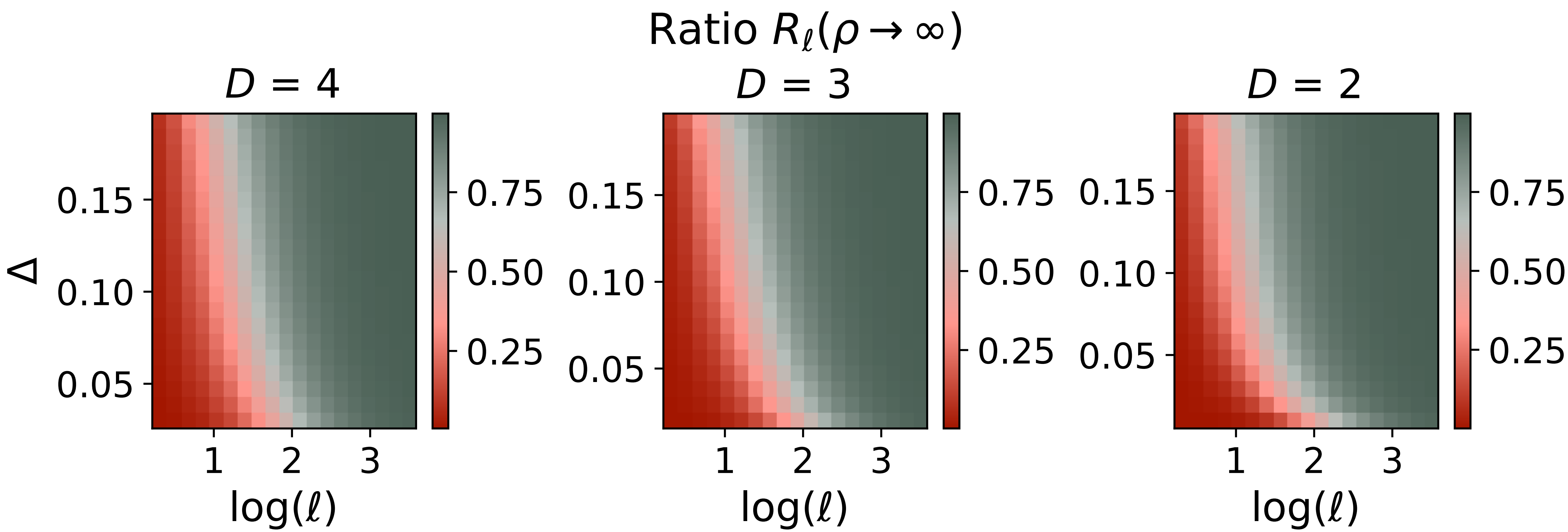
$$d_\ell = \frac{(2\ell + D - 2)\,(\ell + D - 3)!}{\ell!(D-2)!}$$

$$d_0 = 1, \quad d_1 = D$$

does not remove the zero eigenvalues!

Quantum Fluctuations

- Ratio $R_\ell(\rho \to \infty)$
  - < 0 for $\ell = 0 \longrightarrow$ single negative eigenvalue
  - $= 0$ for $\ell = 1 \longrightarrow$ zero eigenvalues
  - $\to 1$ for $\ell \gg 1 \longrightarrow \mathcal{O}_\ell \approx \mathcal{O}_{\mathrm{FV},\ell}$

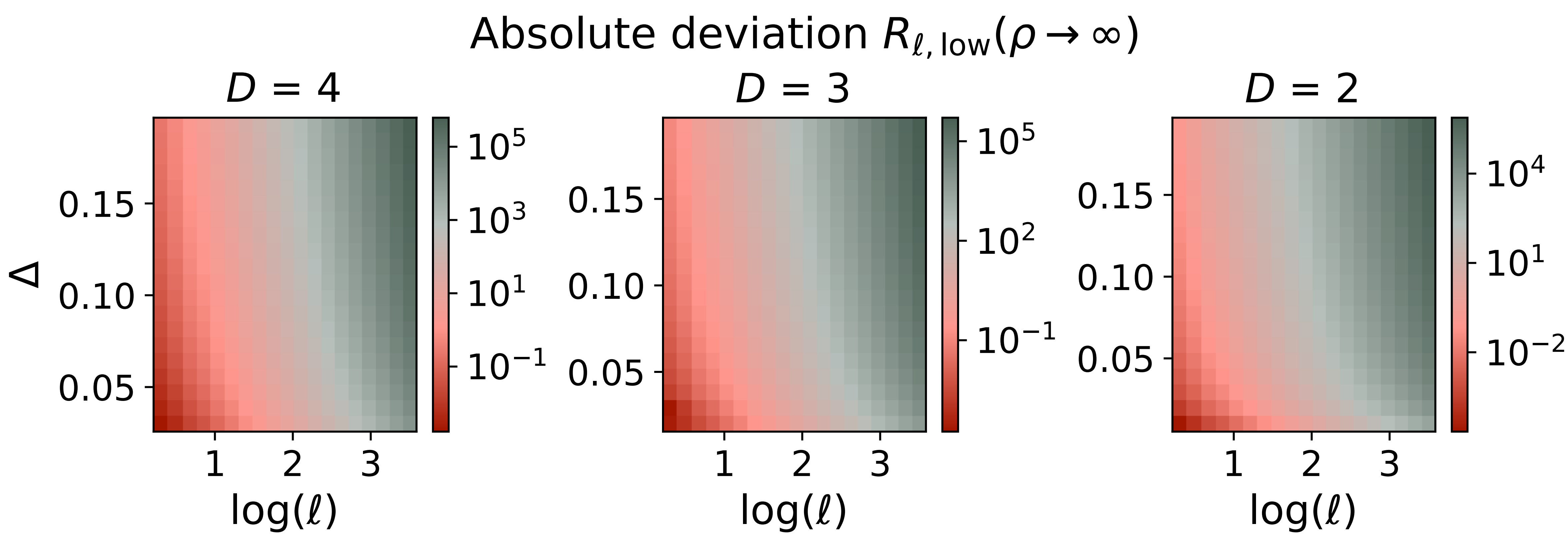


Low Multipoles

- Low multipoles $\nu < 1/\Delta$:

$$R_{\ell,\mathrm{low}}(\rho \to \infty) = \Delta^2 e^{D-1} \frac{3}{4} \frac{\left(\ell - 1\right)\left(\ell + D - 1\right)}{(D-1)^2}$$ of order $\mathcal{O}(\Delta^2)$

- Correct behaviour for $\ell = 0,1$
- Does not converge to 1 for $\ell \gg 1$



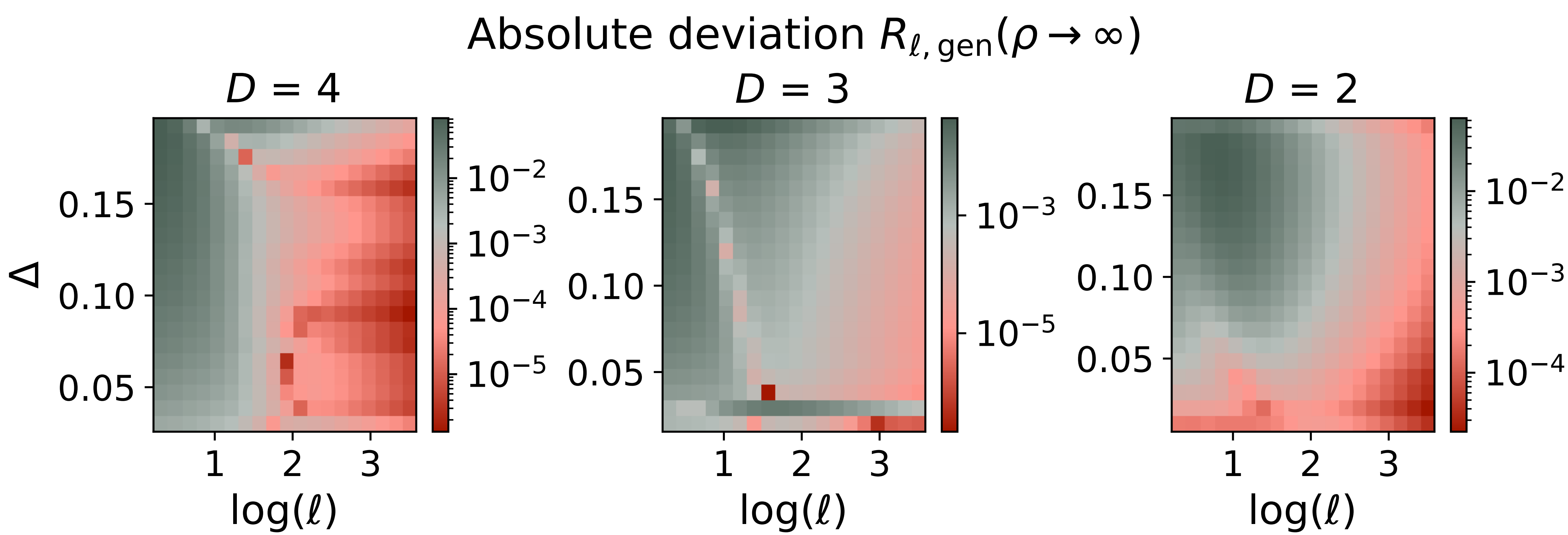
General Multipoles

- New multipole notation: $\nu = \ell + D/2 - 1$
- Treat $\nu\Delta$ as of order $\mathcal{O}(\Delta^0)$

$$R_{\nu,\mathrm{gen}}(\rho \to \infty) = \frac{\left(k_\nu - 1\right)\left(2k_\nu - 1\right)}{\left(k_\nu + 1\right)\left(2k_\nu + 1\right)} e^{3r_0\left(k_\nu - \sqrt{k_\nu^2 - 1}\right)}, \quad k_\nu^2 = 1 + \frac{\Delta^2\nu^2}{r_0^2}$$

of order $\mathcal{O}(\Delta^0)$

- Does converge to 1 for $\ell \gg 1$
- Does not recover the low multipole result



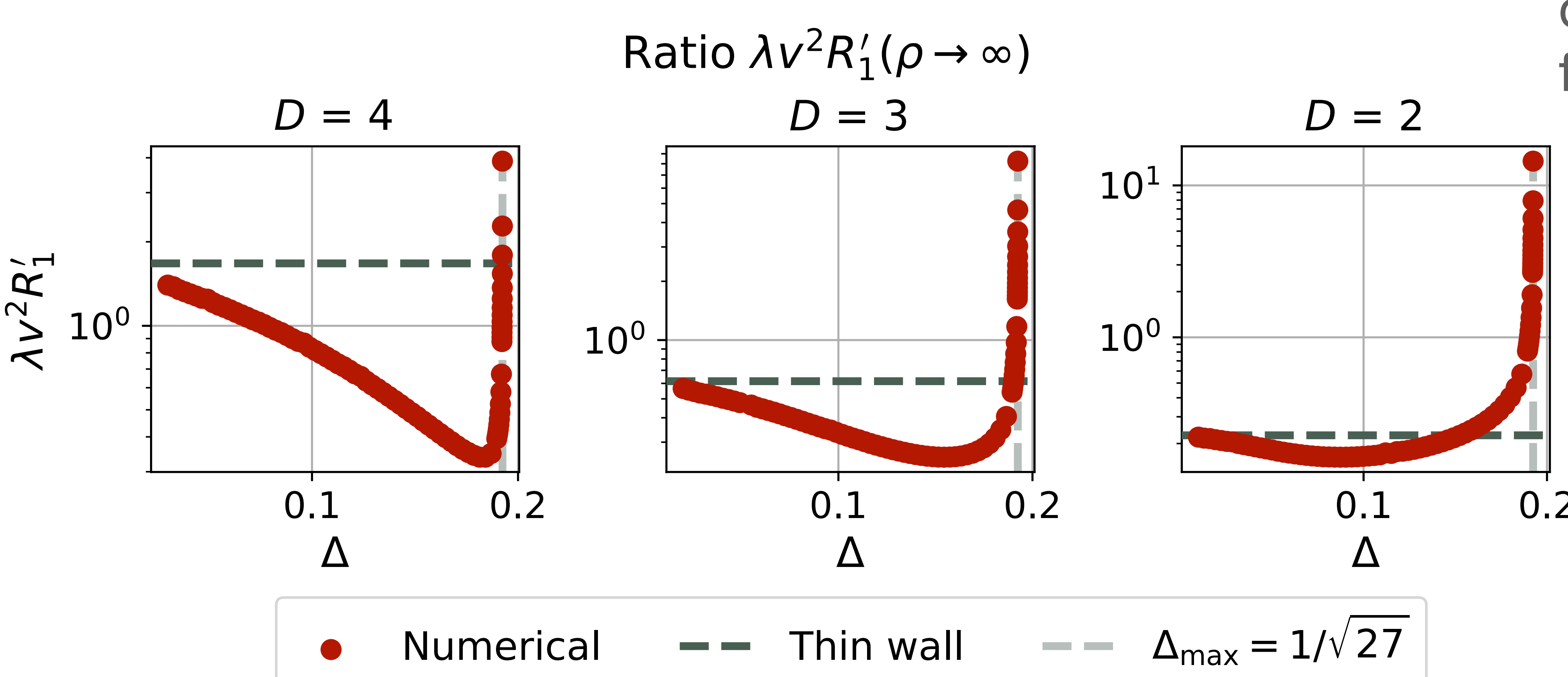
Removal of Zero Eigenvalues

- Using Gel'fand Yaglom does not account for removal of zero eigenvalues at $\ell = 1$
- Add dimensionful μ_ε^2 : $\left(\mathcal{O}_1 + \mu_\varepsilon^2\right)\psi_1^\varepsilon = 0 \to R_1^\varepsilon = \psi_1^\varepsilon/\psi_{\mathrm{FV},1}$

$$R_1'(\rho \to \infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2} R_1^\varepsilon(\rho \to \infty) = \frac{1}{12}\frac{e^{D-1}}{\lambda v^2}$$

reduced dimensionality

diverges for $\Delta_{\max}$



Renormalized Ratio of Functional Determinants

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right|^{-1/2} = \left(\left|R_0\right| R_1'^{D} \prod_{\ell=2}^{\infty} \frac{\det \mathcal{O}_\ell}{\det \mathcal{O}_{\mathrm{FV},\ell}}\right)^{-1/2}$$

- In thin-wall limit the ratio is dominated by high multipoles

$$\ln\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right| \longrightarrow D\left[\ln \tilde{R}_1' - \ln\left(\lambda v^2\right)\right] + \sum_{\nu=\nu_0}^{\infty} d_\nu \ln R_\nu$$

$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right)$

$\tilde{R}_1' = e^{D-1}/12$

$\nu_0 = D/2 - 1$

- The sum diverges:

$$\sum_{\nu \gg 1} d_\nu \ln R_\nu \approx \frac{3r_0\left(2-r_0\right)}{(D-2)!} \sum_{\nu \gg 1} \nu^{D-2}\left[\frac{1}{\nu} - \frac{1}{\nu^3}\left(\frac{r_0}{2\Delta}\right)^2\right]$$

Both terms diverge in $D=4$, only the first term diverges in $D=2{,}3$

→ Apply MS scheme

$$\sum_{\nu=\nu_0}^{\infty} d_\nu \ln R_\nu \to \sum_{\nu=\nu_0}^{\infty} d_\nu \left(\ln R_\nu - \ln R_\nu^a\right) + a_i \tilde{I}_i$$

Added in even dimensions, with ε and μ dependence

Renormalized Ratio of Functional Determinants

$$-S_R - S_{\rm ct} - \frac{1}{2}\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right)\Bigg|_{D=4} \approx -S - \frac{1}{\Delta^3}\frac{27-2\sqrt{3}\pi}{96}$$

$$-S_R - S_{\rm ct} - \frac{1}{2}\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right)\Bigg|_{D=3} \approx -S - \frac{1}{\Delta^2}\frac{20+9\ln 3}{54}$$

$$-S_R - S_{\rm ct} - \frac{1}{2}\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right)\Bigg|_{D=2} \approx -S - \frac{1}{\Delta}\frac{\sqrt{3}\pi-18}{18}$$

add $\tilde{R}_1'$ contribution →

$$\frac{\Gamma}{V} \propto \exp\left[-S - \frac{1}{2}\Sigma_D^f\right]$$

- The finite contribution of quantum fluctuations in the thin-wall limit:

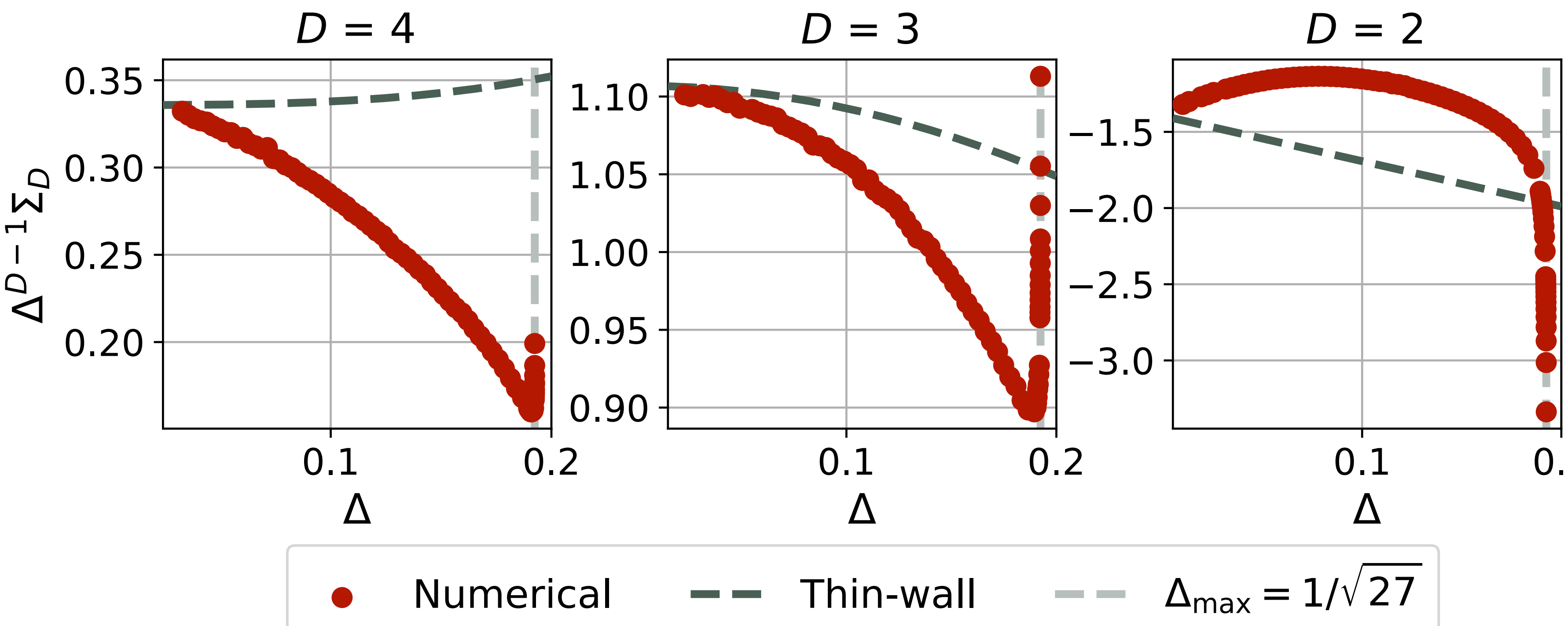
$$\Sigma_4^f = \frac{1}{\Delta^3}\frac{27-2\sqrt{3}\pi}{48} + 12 - 4\ln 12$$

$$\Sigma_3^f = \frac{1}{\Delta^2}\frac{20+9\ln 3}{27} + 6 - 3\ln 12$$

$$\Sigma_2^f = \frac{1}{\Delta}\frac{\sqrt{3}\pi-18}{9} + 2 - 2\ln 12$$

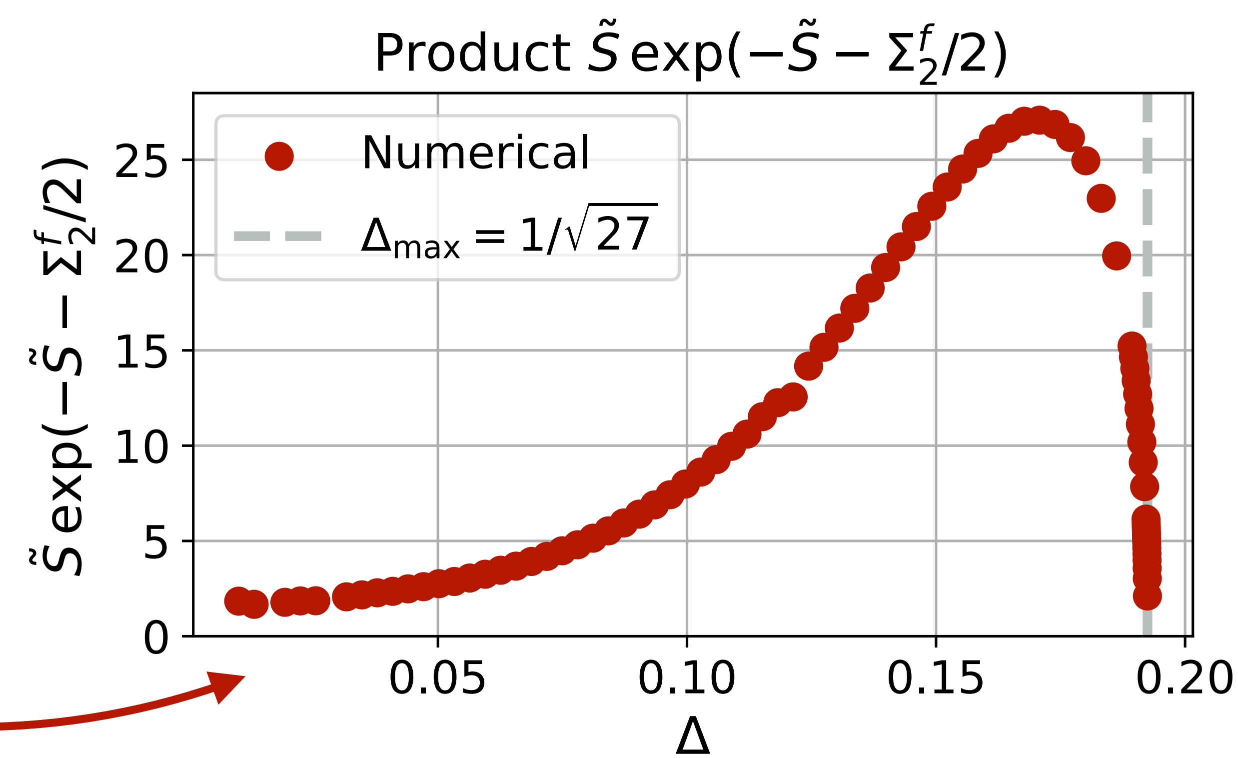
Renormalized Ratio of Functional Determinants

- Comparing the thin-wall and numerical results:



- Divergence to positive values in $D = 4{,}3$
- Divergence to negative values in $D = 2$

decay rate diverges?



no, the behaviour is as expected

Decay Rate

$$\frac{\Gamma}{V} \approx \left(\frac{S_R}{2\pi}\lambda_0 v_0^2\right)^{D/2} \exp\left(-S - \frac{1}{2}\Sigma_D^f\right)$$

- $\Delta = 0$: $S \to \infty$ and $\Sigma_D^f \to \pm\infty$
- $\Delta = \Delta_{\max}$: $S \to 0$ and $\Sigma_D^f \to \pm\infty$

$\longrightarrow$ $\frac{\Gamma}{V} \to 0$

- Renormalization scale dependence in even dimensions

$$S_R \approx \begin{cases} S\left[1 + \frac{9\lambda_0}{(4\pi)^2}\ln\frac{\mu}{\mu_0}\right], & \text{for } D = 4 \\ S\left[1 + \frac{15}{4\pi v_0^2}\ln\frac{\mu}{\mu_0}\right], & \text{for } D = 2 \end{cases}$$

Conclusion

- We explored false vacuum decay rate at one loop in various dimensions
- Found bounce solution and its action, which had to be renormalized
- Quantum fluctuations around the bounce solution give contribution to the decay rate through a ratio of functional determinants
- The decay rate vanishes in both limits, as expected
- The renormalization scale dependance has to be eliminated in even dimensions (higher order corrections)
- Future work could include additional scalars, would-be Goldstones, gauge bosons and fermions

Additional slides

Thin-Wall Limit Expansions

- Bounce solution:

$$\varphi_0(z) = \tanh\left(\frac{z}{2}\right),$$

$$\varphi_1(z) = -1,$$

$$\varphi_2(z) = \frac{3}{4(D-1)\cosh^2(z/2)}\left\{\left[2 - D - 2\left(4+\cosh z\right)\ln\left(1+e^z\right)\right]\sinh z - z\left[D - e^z\left(4+\sinh z\right)\right] + 3\left[\mathrm{Li}_2\left(-e^z\right) - \mathrm{Li}_2\left(-e^{-z}\right)\right]\right\}$$

- Bounce radius:

$$r_0 = \frac{D-1}{3}, \qquad r_1 = 0, \qquad r_2 = \frac{6\pi^2 - 40 + D\left(26 - 4D - 3\pi^2\right)}{3(D-1)}$$

- Bounce action:

$$\tilde{S}_0 = \frac{\Omega_D}{\Delta^{D-1}}\left(\frac{D-1}{3}\right)^{D-1}\frac{2}{3D},$$

$$\tilde{S}_2 = \frac{\Omega_D}{\Delta^{D-1}}\frac{-8D^2 + \left(25 - 3\pi^2\right)D + 1}{2(D-1)},$$

$$\tilde{S}_4 = \frac{\Omega_D}{\Delta^{D-1}}\frac{1}{40(D-1)^3}\left[320D^5 + 80D^4\left(3\pi^2 - 49\right) - 3D^3\left(550\pi^2 + 3\pi^4 - 6185\right) + 5D^2\left(426\pi^2 + 45\pi^4 - 648\zeta(3) - 7843\right) + D\left(3240\zeta(3) + 30635 + 360\pi^2 - 414\pi^4\right) + 105\right]$$

Gel'fand Yaglom Theorem

- Contribution of quantum fluctuations:

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-1/2} = \left|\prod_{\ell=0}^{\infty}\frac{\det' \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}}\right|^{-1/2}$$

$$\mathcal{O}_\ell = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\ell\left(\ell + D - 2\right)}{\rho^2} + m_{\rm B}^2,$$

$$\mathcal{O}_{{\rm FV},\ell} = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\ell\left(\ell + D - 2\right)}{\rho^2} + m_{\rm FV}^2$$

- Gel'fand Yaglom:

$$\mathcal{O}_\ell\,\psi_\ell(\rho) = 0$$

$$\mathcal{O}_{{\rm FV},\ell}\,\psi_{{\rm FV},\ell}(\rho) = 0$$

$$\psi_{({\rm FV}),\ell}(\rho \sim 0) = \rho^\ell$$

→

$$\frac{\det \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}} = \lim_{\rho\to\infty}\left(\frac{\psi_\ell(\rho)}{\psi_{{\rm FV},\ell}(\rho)}\right)^{d_\ell} = R_\ell(\rho\to\infty)^{d_\ell}$$

$$\psi_{{\rm FV},\ell}(\rho\sim 0) = \rho^\ell,\quad R_\ell(\rho=0)=1,\quad R'_\ell(\rho=0)=0$$

Gel'fand Yaglom Theorem

- Contribution of quantum fluctuations:

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-1/2} = \left|\prod_{\ell=0}^{\infty}\frac{\det' \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}}\right|^{-1/2}$$

$$\mathcal{O}_\ell = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\ell\left(\ell+D-2\right)}{\rho^2} + m_{\rm B}^2,$$

$$\mathcal{O}_{{\rm FV},\ell} = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\ell\left(\ell+D-2\right)}{\rho^2} + m_{\rm FV}^2$$

- Gel'fand Yaglom:

$$\frac{\det \mathcal{O}_\ell}{\det \mathcal{O}_{{\rm FV},\ell}} = R_\ell(\rho\to\infty)^{d_\ell}$$

$$\psi''_{{\rm FV},\ell} + \frac{D-1}{\rho}\psi'_{{\rm FV},\ell} = \left[\frac{\ell\left(\ell+D-2\right)}{\rho^2} + m_{\rm FV}^2\right]\psi_{{\rm FV},\ell}$$

$$R''_\ell + 2\left(\frac{\psi'_{{\rm FV},\ell}}{\psi_{{\rm FV},\ell}}\right)R'_\ell = \left(m_{\rm B}^2 - m_{\rm FV}^2\right)R_\ell$$

$$\psi_{{\rm FV},\ell}(\rho\sim 0) = \rho^\ell, \quad R_\ell(\rho=0)=1, \quad R'_\ell(\rho=0)=0$$

Renormalized Ratio in Thin-Wall Limit

- Renormalized ratio:

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right)\Bigg|_{D=4} = \sum_{\nu=1}^{\infty} \nu^2 \left(\ln R_\nu - \frac{1}{2\nu} I_1 + \frac{1}{8\nu^3} I_2\right) - \frac{1}{8}\tilde{I}_2$$

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right)\Bigg|_{D=3} = \sum_{\nu=1/2}^{\infty} 2\nu \left(\ln R_\nu - \frac{1}{2\nu} I_1\right)$$

$$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\mathrm{FV}}}\right)\Bigg|_{D=2} = 2\ln R_0 + \sum_{\nu=1}^{\infty} 2 \left(\ln R_\nu - \frac{1}{2\nu} I_1\right) + \tilde{I}_1$$

dependent on ε and μ

- UV integrals:

$$I_1 = \int_0^\infty \mathrm{d}\rho\, \rho\, \left(m_{\mathrm{B}}^2 - m_{\mathrm{FV}}^2\right) \approx -3\,(2 - r_0)\left(\frac{r_0}{\Delta}\right),$$

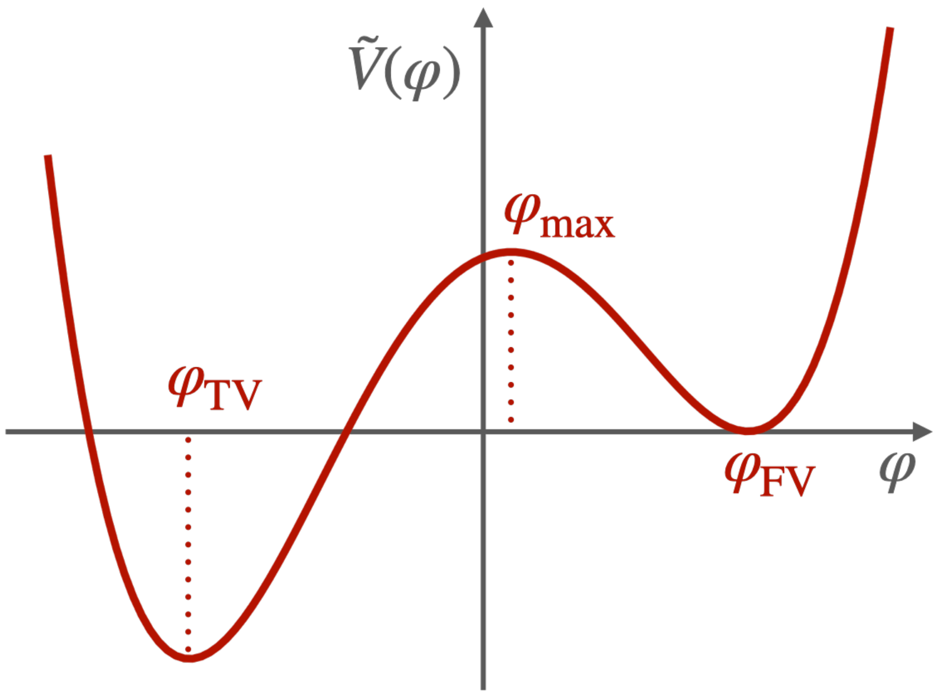
$$I_2 = \int_0^\infty \mathrm{d}\rho\, \rho^3 \left[\left(m_{\mathrm{B}}^2\right)^2 - \left(m_{\mathrm{FV}}^2\right)^2\right] \approx -3\,(2 - r_0)\left(\frac{r_0}{\Delta}\right)^3,$$

$$\tilde{I}_1 = \int_0^\infty \mathrm{d}\rho\, \rho\, \left(m_{\mathrm{B}}^2 - m_{\mathrm{FV}}^2\right)\left[\frac{1}{\varepsilon} + \gamma + \ln\left(\frac{\mu\rho}{2}\right)\right] \overset{D=2}{\approx} -\frac{1}{6\Delta} + \left[\frac{1}{\varepsilon} + \gamma + \ln\frac{\mu}{6\sqrt{\lambda_0}v_0\Delta}\right] I_1,$$

$$\tilde{I}_2 = \int_0^\infty \mathrm{d}\rho\, \rho^3 \left[\left(m_{\mathrm{B}}^2\right)^2 - \left(m_{\mathrm{FV}}^2\right)^2\right]\left[\frac{1}{\varepsilon} + \gamma + 1 + \ln\left(\frac{\mu\rho}{2}\right)\right] \overset{D=4}{\approx} I_2\left[\frac{1}{\varepsilon} + \gamma + \frac{5}{4} + \ln\frac{\mu}{2\sqrt{\lambda_0}v_0\Delta}\right]$$

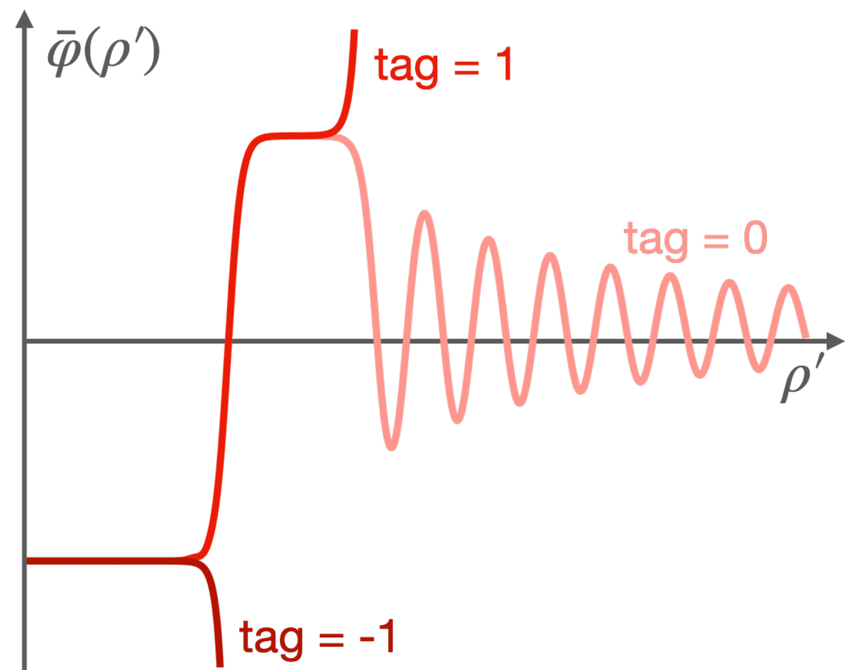
Numerical Methods

- Finding bounce solution using shooting method:



$$\varphi'' + \frac{D-1}{\rho'}\varphi' - \frac{\partial \tilde{V}}{\partial \varphi} = 0$$

$$\bar{\varphi}'(0) = \bar{\varphi}'(\infty) = 0,$$
$$\bar{\varphi}(0) = \varphi_{\rm in},\ \bar{\varphi}(\infty) = \varphi_{\rm FV}$$



- Finite sum:

$$\Sigma_4^f = \ln|R_1| - \frac{1}{2}I_1 + \frac{1}{8}I_2 + 4\left(\ln R_2' - \frac{1}{4}I_1 + \frac{1}{64}I_2\right) + \sum_{\nu=3}^{\infty}\nu^2\left(\ln R_\nu - \frac{1}{2\nu}I_1 + \frac{1}{8\nu^3}I_2\right) - \underline{\frac{1}{8}\tilde{I}_2^R},$$

$$\Sigma_3^f = \ln|R_{1/2}| - I_1 + 3\left(\ln R_{3/2}' - \frac{1}{3}I_1\right) + \sum_{\nu=5/2}^{\infty}2\nu\left(\ln R_\nu - \frac{1}{2\nu}I_1\right),$$

$$\Sigma_2^f = \ln|R_0| + 2\left(\ln R_1' - \frac{1}{2}I_1\right) + \sum_{\nu=2}^{\infty}2\left(\ln R_\nu - \frac{1}{2\nu}I_1\right) + \underline{\tilde{I}_1^R}$$

without ε term and $\mu = 1$

Numerical Methods

- Finding ratio R_ν :

$$\overbrace{\left\{\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+2\left[\frac{1}{2\rho}+\sqrt{m_{\mathrm{FV}}^2}\,\frac{I_\nu'\left(\sqrt{m_{\mathrm{FV}}^2}\rho\right)}{I_\nu\left(\sqrt{m_{\mathrm{FV}}^2}\rho\right)}\right]\frac{\mathrm{d}}{\mathrm{d}\rho}-\left(m_{\mathrm{B}}^2-m_{\mathrm{FV}}^2\right)\right\}}^{\mathcal{O}_{R,\nu}} R_\nu(\rho)=0$$

initial conditions:

$$R_\nu(\rho=0)=1,\quad R_\nu'(\rho=0)=0$$

modified Bessel function of the first type I_ν comes from the analytic solution:

$$\psi_{\mathrm{FV},\nu}(\rho)=c_\nu\sqrt{\rho}\,I_\nu\left(\sqrt{m_{\mathrm{FV}}^2}\rho\right)$$

for low values of z:

$$I_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}$$

- Finding reduced ratio $R_{D/2}'$:

$$\left(\mathcal{O}_{R,D/2}-\mu_\varepsilon^2\right)\left(R_{D/2}(\rho)+\mu_\varepsilon^2R_{D/2}'(\rho)\right)=0 \longrightarrow \mathcal{O}_{R,D/2}R_{D/2}'(\rho)=R_{D/2}(\rho)$$

Dimensional Analysis and Dimensionless Quantities

- Known dimensionalities: $[x^\mu] = -1, \quad [\mathrm{d}x^\mu] = -1, \quad [\partial_\mu] = 1, \quad [S] = 0$
- We get:

$$
\begin{aligned}
[S] = [\mathrm{d}^D x \partial_\mu \phi \partial^\mu \phi] = D[\mathrm{d}x] + 2[\partial_\mu] + 2[\phi] \equiv 0 \quad &\Rightarrow \quad [\phi] = \frac{1}{2}(D-2), \\
[S] = [\mathrm{d}^D x \lambda \phi^4] = D[\mathrm{d}x] + 4[\phi] + [\lambda] \equiv 0 \quad &\Rightarrow \quad [\lambda] = 4 - D, \\
[S] = [\mathrm{d}^D x \lambda v^4] = D[\mathrm{d}x] + [\lambda] + 4[v] \equiv 0 \quad &\Rightarrow \quad [v] = \frac{1}{2}(D-2), \\
[S] = [\mathrm{d}^D x \lambda v^4 \Delta] = D[\mathrm{d}x] + [\lambda] + 4[v] + [\Delta] \equiv 0 \quad &\Rightarrow \quad [\Delta] = 0
\end{aligned}
$$

- In thermal ($D = 4$) field theory time dimension becomes compactified, reducing the effective dimensionality of the theory to $D = 3$

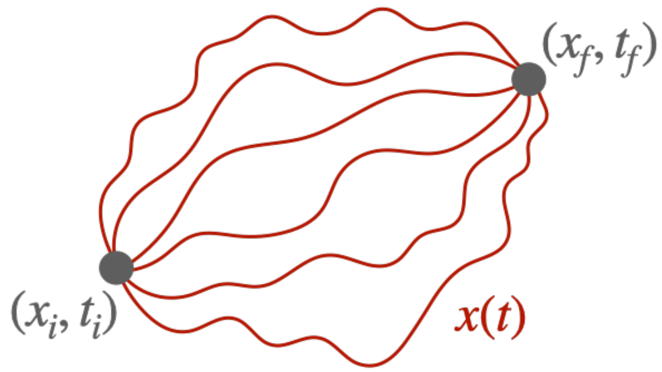
$$\longrightarrow [S] = [\mathrm{d}^3 x \lambda v^4 \Delta] = 1 \xrightarrow{\text{divide } S \text{ by } T} [S/T] = [S] - [T] = 0$$

- Decay rate and ratio of determinants:

$$[\Gamma] = -[\tau] = 1 \longrightarrow [\Gamma/V] = D = \left[\left| \frac{\det'\left[-\partial^2 + m_\mathrm{B}^2\right]}{\det\left[-\partial^2 + m_\mathrm{FV}^2\right]} \right|^{-1/2} \right]$$

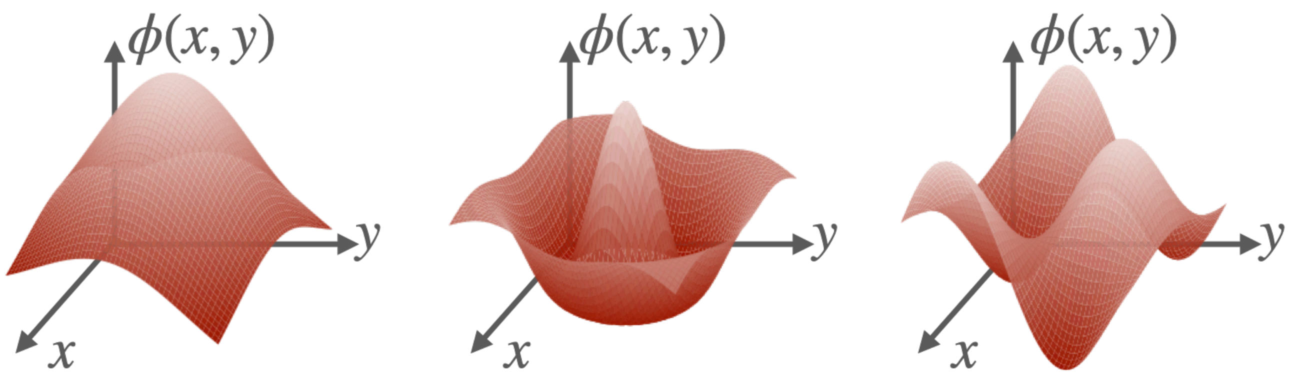
Path Integral in Euclidean Space

- Path integral in quantum mechanics (QM):



$$G(x_f, t_f; x_i, t_i) = N \sum_{\text{all paths } x(t)} e^{\mathrm{i}S[x(t)]/\hbar} = N \int \mathscr{D}x\, e^{\mathrm{i}S[x(t)]/\hbar}$$

- Path integral in scalar quantum field theory (SQFT):



$$G(\phi_f, t_f; \phi_i, t_i) = \int \mathscr{D}\phi\, e^{\mathrm{i}S[\phi]/\hbar}$$

- Euclidean space $t \to -\mathrm{i}t_E$:

QM: $$G_E(x_f, t_f; x_i, t_i) = N \int \mathscr{D}x\, e^{-S_E[x(t)]/\hbar}, \quad S_E[x(t)] = \int \mathrm{d}t_E\, L_E$$

SQFT: $$G_E(\phi_f, t_f; \phi_i, t_i) = \int \mathscr{D}\phi\, e^{-S_E[\phi]/\hbar}, \quad S_E[\phi] = \int \mathrm{d}^4 x_E\, \mathscr{L}_E$$