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Regularization of Functional Determinants of Radial Operators via Heat Kernel Coefficients

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Contents

Introduction

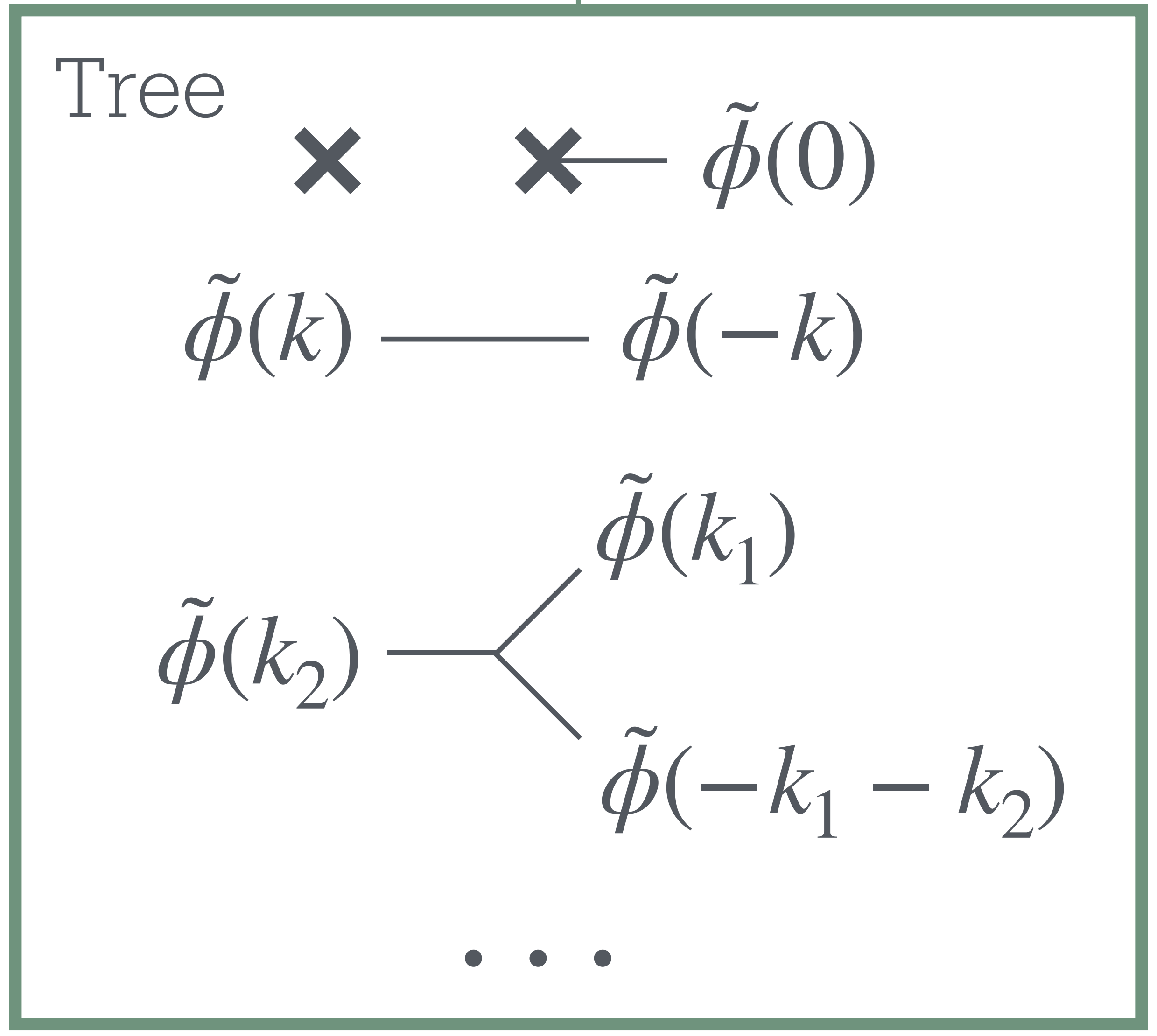
Functional determinant

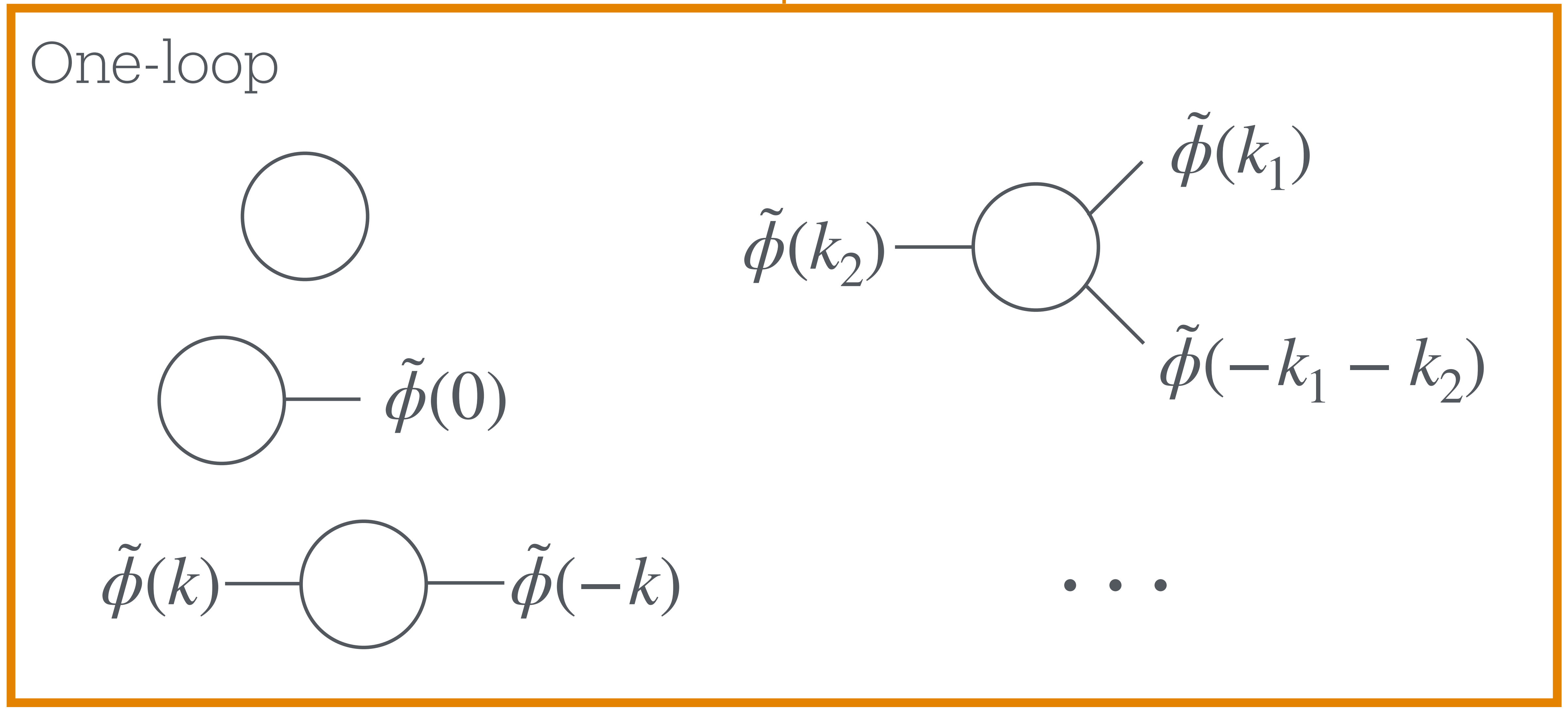
Effective action (Euclidean)

$$\Gamma[\phi] = S[\phi] + \ln \int \mathscr{D}\varphi e^{-S[\phi+\varphi]+\frac{d\Gamma}{d\phi}\varphi}$$

$$= S[\phi] - \frac{1}{2}\ln\det[-\partial^2 + m^2(x)] + \mathcal{O}(\hbar^2)$$

$$m^2(x) = V''(\phi(x))$$





Functional determinant

Effective potential (Euclidean)

$$V_{\text{eff}}(\phi) = \Gamma[\phi = \text{const.}]$$

const. $\hat{m}^2 = V''(\phi)$

$$= V(\phi) - \frac{1}{2}\ln\det[-\partial^2 + \hat{m}^2] + \mathcal{O}(\hbar^2)$$

Calculation

$$\ln\det[-\partial^2 + \hat{m}^2] = \operatorname{tr}\ln[-\partial^2 + \hat{m}^2]$$

Eigenstate: e^{ikx}

$$= \mathscr{V}\int\frac{d^Dk}{(2\pi)^D}\ln[k^2 + \hat{m}^2] \quad \text{(Divergent)}$$

Spacetime volume

Functional determinant

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Spacetime volume

$$\to \mathscr{V}\mu^{2\varepsilon}\int \frac{d^{D-2\varepsilon}k}{(2\pi)^{D-2\varepsilon}}\ln[k^2 + \hat{m}^2] \quad \text{Dimensional regularization}$$

Functional determinant

Effective potential (Euclidean)

$$V_{\text{eff}}(\phi) = \Gamma[\phi = \text{const.}]$$

const. $\quad \hat{m}^2 = V''(\phi)$

$$= V(\phi) - \frac{1}{2} \ln \det[-\partial^2 + \hat{m}^2] + \mathcal{O}(\hbar^2)$$

Calculation

$$\ln \det[-\partial^2 + \hat{m}^2] = \operatorname{tr} \ln[-\partial^2 + \hat{m}^2]$$

Eigenstate: e^{ikx}

$$= \mathscr{V} \int \frac{d^D k}{(2\pi)^D} \ln[k^2 + \hat{m}^2] \quad \text{(Divergent)}$$

Spacetime volume

$$\to \mathscr{V} \mu^{2\varepsilon} \int \frac{d^{D-2\varepsilon} k}{(2\pi)^{D-2\varepsilon}} \ln[k^2 + \hat{m}^2] \quad \text{Dimensional regularization}$$

$$\to -\mathscr{V} \frac{d}{ds} \int \frac{d^D k}{(2\pi)^D} [k^2 + \hat{m}^2]^{-s} \quad \text{Zeta function regularization}$$

Functional determinant

Effective action (Euclidean)

$$\Gamma[\phi] = S[\phi] + \ln \int \mathcal{D}\varphi e^{-S[\phi+\varphi]+\frac{d\Gamma}{d\phi}\varphi}$$

$$= S[\phi] - \frac{1}{2} \ln \det[-\partial^2 + m^2(x)] + \mathcal{O}(\hbar^2)$$

Calculation

$$\ln \det[-\partial^2 + m^2(x)] = \mathrm{tr} \ln[-\partial^2 + m^2(x)]$$

$$= \sum_i \ln \lambda_i \quad \text{(Divergent)}$$

Eigenstate: ψ_i

$$[-\partial^2 + m^2(x)]\psi_i = \lambda_i \psi_i$$

How do we regularize the UV divergence?

-> Subtract-and-add-back technique (spherically symmetric case)

Baacke-Lavrelashvili '04 (Feynman diagram)

Dunne-Kirsten '06 (WKB)

Hur-Min '08 (Radial WKB) [Dunne-Hur-Lee-Min '04 for QCD instantons]

Shoji-Yamaguchi '25 (Heat kernel expansion) ←

Subtract-and-add-back technique

Setup

Spherical symmetry

Eigenstates

$$[-\partial^2 + m^2(|x|)]\psi_{\nu\chi i} = \lambda_{\nu i}\psi_{\nu\chi i}$$

$$\psi_{\nu\chi i}(x) = \frac{f_{\nu\chi i}(|x|)}{|x|^{\frac{D}{2}-1}} Y_{\nu\chi}(\hat{x})$$

Radial part

$$\left[-\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2} + m^2(r)\right] f_{\nu\chi i}(r) = \lambda_{\nu i} f_{\nu\chi i}$$

Angular part

$Y_{\nu\chi}(\hat{x})$: Hyper-spherical function

$\nu = \frac{D}{2} - 1, \frac{D}{2}, \cdots, \infty$: (Shifted) angular momentum

$\chi = 1, 2, \cdots, d_\nu$: Label for degenerate states

$d_\nu \sim \nu^{D-2}$

Ratio of determinants

$$\ln \frac{\det[-\partial^2 + m^2(|x|)]}{\det[-\partial^2 + \hat{m}^2]} = \sum_\nu d_\nu \ln \frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}}$$

$m^2(r) \to \hat{m}^2$ fast enough

The ratio is often an important quantity, e.g. prefactor of vacuum decay rate

We are not interested in the Casimir energy that diverges proportional to the volume

Subtract-and-add-back technique

The sum over i can be taken using the Gelfand-Yaglom theorem and is finite

$$\ln \frac{\det[-\partial^2 + m^2(|x|)]}{\det[-\partial^2 + \hat{m}^2]} = \sum_\nu \left[d_\nu \ln \frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}} - s_\nu \right] + S$$

1. The sum should become finite

(Converges as #nu increases)

$$S = \left[\sum_\nu s_\nu \right]_{\text{reg}}$$

Regularized sum of the reference series

2. The regularized value should be known

What is a "better" choice?

$$\left| d_\nu \ln \frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}} - s_\nu \right| < C\nu^{-n}$$

$$n \geq 2$$

"Larger n, Smaller C"

→ Enables early truncation of the series

The easier, the better

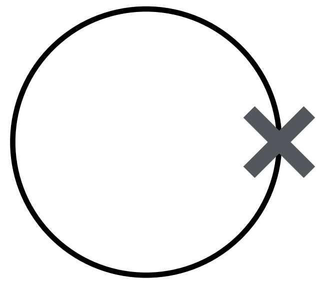
Feynman diagrammatic approach

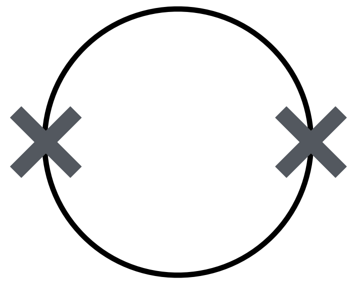
Baacke-Lavrelashvili '04

$$\ln \frac{\det[-\partial^2 + m^2]}{\det[-\partial^2 + \hat{m}^2]} = \mathrm{tr} \ln \left[1 + \frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \right]$$

$$= \mathrm{tr} \left[\frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \right] - \frac{1}{2} \mathrm{tr} \left[\frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \right] + \cdots$$

$$\delta m^2 = m^2 - \hat{m}^2$$





The first few terms are divergent

$$s_\nu = \sum_{a=1}^{a_{\max}} \frac{(-1)^{a-1}}{a} d_\nu \mathrm{tr} \left[P_\nu \left(\frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \right)^a \right]$$

$$\langle x | P_{\nu(\chi)} | x' \rangle = \delta(|x| - |x'|) Y^*_{\nu\chi}(x) Y_{\nu\chi}(x')$$

This can be computed using the Gelfand-Yaglom theorem

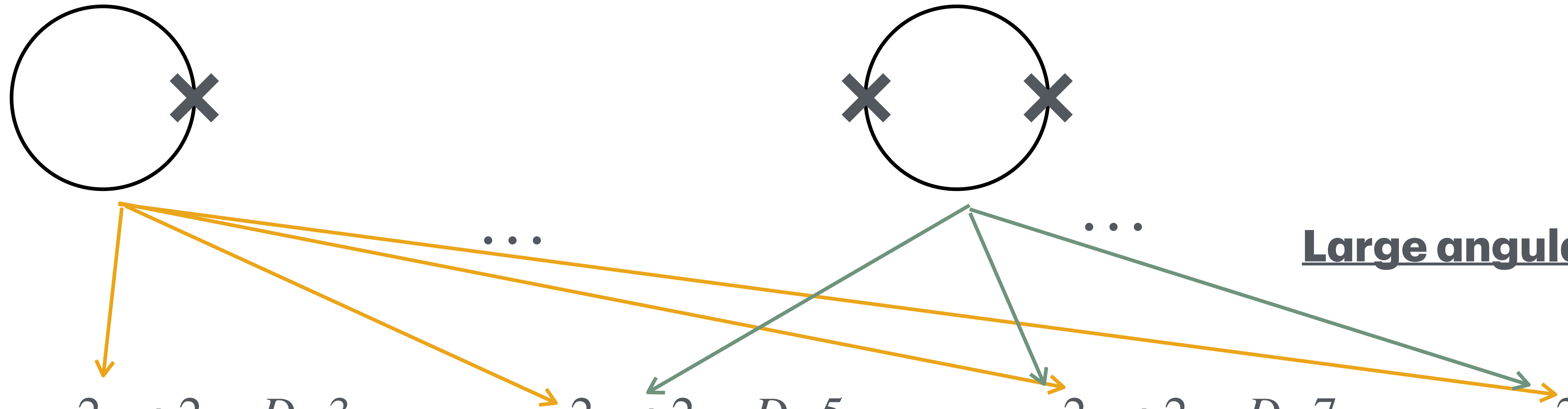
$$S = \sum_{a=1}^{a_{\max}} \frac{(-1)^{a-1}}{a} \mathrm{tr} \left[\left(\frac{1}{-\partial^2 + \hat{m}^2} \delta m^2 \right)^a \right]$$

Usual Feynman diagrammatic computation
(Dimensional Regularization)

WKB approach

Dunne-Kirsten '06

$$\ln \frac{\det[-\partial^2 + m^2]}{\det[-\partial^2 + \hat{m}^2]} = \mathrm{tr}\left[\frac{1}{-\partial^2 + \hat{m}^2}\delta m^2\right] - \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + \hat{m}^2}\delta m^2 \frac{1}{-\partial^2 + \hat{m}^2}\delta m^2\right] + \cdots$$



Large angular momentum expansion

$$= \sum_{\nu} \left[C_1[m^2, \hat{m}^2]\nu^{D-3} + C_2[m^2, \hat{m}^2]\nu^{D-5} + C_3[m^2, \hat{m}^2]\nu^{D-7} + C_4[m^2, \hat{m}^2]\nu^{D-9} + \cdots\right]$$

$$s_\nu = \sum_{a=1}^{a_{\max}} C_a[m^2, \hat{m}^2]\nu^{D-1-2a}$$

WKB analysis of Gelfand-Yaglom theorem

Zeta function regularization

$$S = \sum_{a=1}^{a_{\max}} \lim_{s\to 0} \frac{d}{ds} C_a[m^2, \hat{m}^2](s)\zeta\left(1 - D + 2a + 2s, \frac{D-2}{2}\right)$$

Problems

Feynman Diagram

$$\ln\frac{\det[-\partial^2+m^2]}{\det[-\partial^2+\hat{m}^2]}=\mathrm{tr}\left[\frac{1}{-\partial^2+\hat{m}^2}\delta m^2\right]-\frac{1}{2}\mathrm{tr}\left[\left(\frac{1}{-\partial^2+\hat{m}^2}\delta m^2\right)^2\right]+\frac{1}{3}\mathrm{tr}\left[\left(\frac{1}{-\partial^2+\hat{m}^2}\delta m^2\right)^3\right]+\cdots$$

$$\int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D}\int\frac{d^Dk_3}{(2\pi)^D}\frac{1}{k_1^2+\hat{m}^2}\widetilde{\delta m^2}(k_2-k_1)\frac{1}{k_2^2+\hat{m}^2}\widetilde{\delta m^2}(k_3-k_2)\frac{1}{k_3^2+\hat{m}^2}\widetilde{\delta m^2}(k_1-k_3)$$

2 Feynman integrals + 2 absolute momenta + 1 angle

WKB

$$\ln\frac{\det[-\partial^2+m^2]}{\det[-\partial^2+\hat{m}^2]}=\sum_\nu\left[C_1[m^2,\hat{m}^2]\nu^{D-3}+C_2[m^2,\hat{m}^2]\nu^{D-5}+C_3[m^2,\hat{m}^2]\nu^{D-7}+C_4[m^2,\hat{m}^2]\nu^{D-9}+\cdots\right]$$

Not known

Problems

1. These can only be used for D<6

2. One cannot systematically improve the convergence

$$\left|d_\nu\ln\frac{\prod_i\lambda_{\nu i}}{\prod_i\hat{\lambda}_{\nu i}}-s_\nu\right|<C\nu^{-n}$$

These problems are solved in Hur-Min '08 (Radial WKB) Shoji-Yamaguchi '25 (Heat kernel expansion)

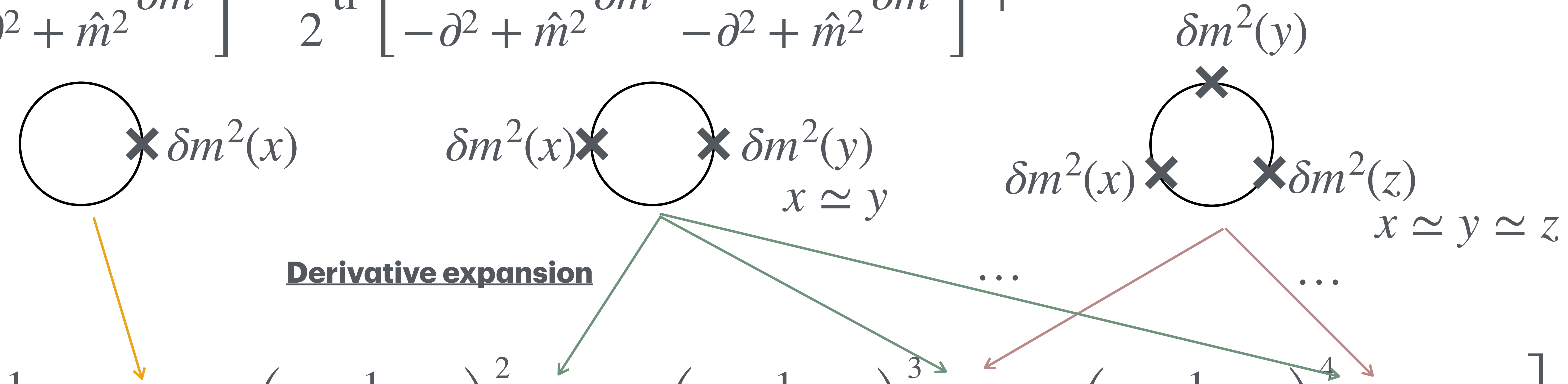
Significantly simple formulas

Heat kernel expansion

HKE approach

YS, M. Yamaguchi '25

$$\ln \frac{\det[-\partial^2 + m^2]}{\det[-\partial^2 + \hat{m}^2]} = \mathrm{tr}\left[\frac{1}{-\partial^2 + \hat{m}^2}\delta m^2\right] - \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + \hat{m}^2}\delta m^2 \frac{1}{-\partial^2 + \hat{m}^2}\delta m^2\right] + \cdots$$



$$= \mathrm{tr}\left[\frac{1}{-\partial^2 + \hat{m}^2}\tilde{C}_1(x) + \left(\frac{1}{-\partial^2 + \hat{m}^2}\right)^2 \tilde{C}_2(x) + \left(\frac{1}{-\partial^2 + \hat{m}^2}\right)^3 \tilde{C}_3(x) + \left(\frac{1}{-\partial^2 + \hat{m}^2}\right)^4 \tilde{C}_4(x) + \cdots\right]$$

$$= \sum_{a=1}^{\infty} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + \hat{m}^2)^a} \int d^D x \tilde{C}_a(x)$$

← The first few terms are divergent

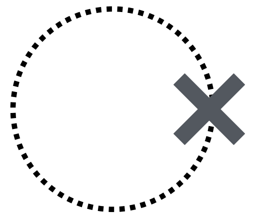
Overall factor

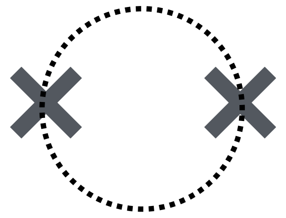
Dimensional regularization

Zeta function regularization

HKE approach (generalization)

$$\ln \frac{\det[-\partial^2 + m^2]}{\det[-\partial^2 + \hat{m}^2]} = \mathrm{tr}\left[\frac{1}{-\partial^2 + z}(m^2 - z)\right] - \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(m^2 - z)\frac{1}{-\partial^2 + z}(m^2 - z)\right] + \cdots$$

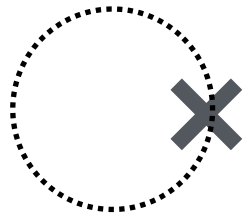


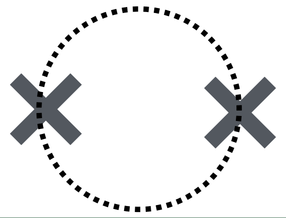


Numerator

$z \geq 0$: Propagator mass

$$-\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(\hat{m}^2 - z)\right] + \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(\hat{m}^2 - z)\frac{1}{-\partial^2 + z}(m^2 - z)\right] - \cdots$$





Denominator

$$= \sum_{a=1}^{\infty} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + z)^a} \int d^D x \, \delta \tilde{C}_a^z(x)$$

HKE approach (generalization)

$$\ln \frac{\det[-\partial^2 + m^2]}{\det[-\partial^2 + \hat{m}^2]} = \mathrm{tr}\left[\frac{1}{-\partial^2 + z}(m^2 - z)\right] - \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(m^2 - z)\frac{1}{-\partial^2 + z}(m^2 - z)\right] + \cdots$$

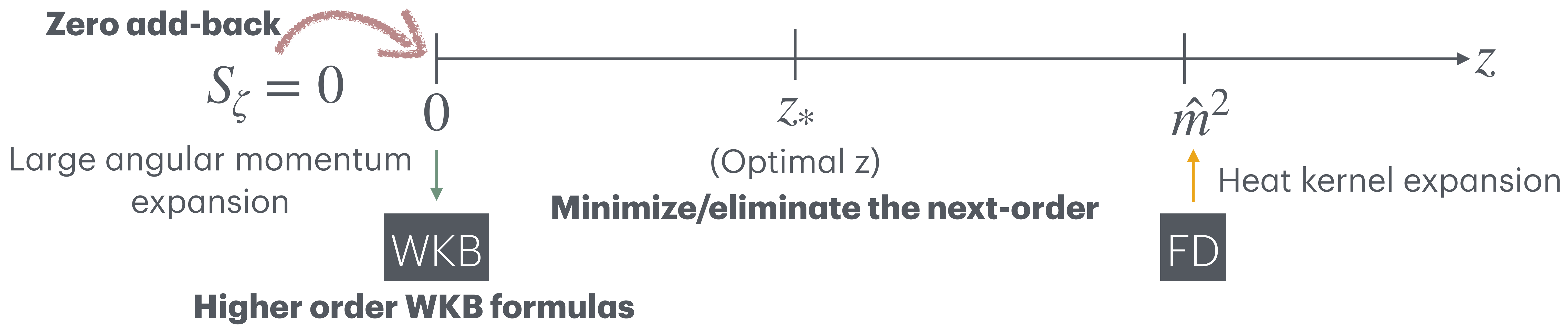
Numerator

$z \geq 0$: Propagator mass

$$-\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(\hat{m}^2 - z)\right] + \frac{1}{2}\mathrm{tr}\left[\frac{1}{-\partial^2 + z}(\hat{m}^2 - z)\frac{1}{-\partial^2 + z}(m^2 - z)\right] - \cdots$$

Denominator

$$= \sum_{a=1}^{\infty} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + z)^a} \int d^D x \, \delta \tilde{C}_a^z(x)$$



HKE formulas

Subtraction

$$s_\nu(z) = -\,d_\nu \sum_{a=1}^{a_{\max}} \int_0^\infty dr\, r \left(B_a^z[m^2 - z](r) - B_a^z[\hat{m}^2 - z](r)\right) \mathcal{I}_{a,\nu}(z,r)$$

Next slide

$z \neq 0$

$$\mathcal{I}_{a,\nu}(z,r) = \left(-\partial_z\right)^{a-1} I_\nu\left(\sqrt{z}r\right) K_\nu\left(\sqrt{z}r\right)$$

Notice that $\partial_z I_\nu(z) = (I_{\nu+1}(z) + I_{\nu-1}(z))/2$, etc.

$z = 0$

$$\mathcal{I}_{a,\nu}(0,r) = \lim_{s\to 0} \frac{d}{ds} \frac{\Gamma\left(a+s-\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(s)} \frac{\Gamma(\nu - a + 1 - s)}{\Gamma(\nu + a + s)} r^{2(a+s-1)}$$

nu-dependence

r-dependence

The r-integral can be done independently of nu

Add-back

$$S(z) = -\sum_{a=1}^{a_{\max}} \int d^D x \left(B_a^z[m^2 - z](|x|) - B_a^z[\hat{m}^2 - z](|x|)\right) \left[\lim_{s\to 0} \frac{d}{ds} \frac{\Gamma(a+s)}{\Gamma(s)} \mu^{2\varepsilon} \int \frac{d^{D-2\varepsilon}k}{(2\pi)^{D-2\varepsilon}} \frac{1}{(k^2+z)^{a+s}}\right]$$

Dimensional/zeta function regularization

$s = 0$, $\varepsilon \to 0$

$\varepsilon = 0$, $s \to 0$

HKE formulas

$$B_a^z[m^2] = B_a^{z=0}[m^2] + \Delta B_a^z[m^2]$$

Heat kernel coefficient + Total derivatives

$$B_1^{z=0}[m^2] = -m^2,$$

$$B_2^{z=0}[m^2] = \frac{1}{2}(m^2)^2,$$

$$B_3^{z=0}[m^2] = -\frac{1}{6}\left[(m^2)^3 + \frac{1}{2}(\partial_\mu m^2)(\partial^\mu m^2)\right],$$

$$B_4^{z=0}[m^2] = \frac{1}{24}\left[(m^2)^4 + 2m^2(\partial_\mu m^2)(\partial^\mu m^2) + \frac{4}{15}(\partial_\mu\partial_\nu m^2)(\partial^\mu\partial^\nu m^2) + \frac{1}{15}(\partial^\mu m^2)(\partial_\mu\partial^2 m^2)\right],$$

$$B_5^{z=0}[m^2] = -\frac{1}{120}\left[(m^2)^5 + 5(m^2)^2(\partial_\mu m^2)(\partial^\mu m^2) + \frac{5}{4}m^2(\partial_\mu\partial_\nu m^2)(\partial^\mu\partial^\nu m^2) + \frac{1}{4}m^2(\partial_\mu m^2)(\partial^2\partial^\mu m^2) + \frac{13}{6}(\partial_\mu m^2)(\partial_\nu m^2)(\partial^\mu\partial^\nu m^2) + \frac{1}{8}(\partial_\mu m^2)(\partial^\mu m^2)(\partial^2 m^2) + \frac{1}{7}(\partial_\mu\partial_\nu\partial_\rho m^2)(\partial^\mu\partial^\nu\partial^\rho m^2) + \frac{5}{56}(\partial_\mu\partial_\nu m^2)(\partial^\mu\partial^\nu\partial^2 m^2) + \frac{1}{112}(\partial_\mu\partial^2 m^2)(\partial^\mu\partial^2 m^2) + \frac{1}{112}(\partial_\mu m^2)(\partial^\mu\partial^4 m^2)\right].$$

$$\Delta B_1^z[m^2] = \Delta B_2^z[m^2] = \Delta B_3^z[m^2] = \Delta B_4^z[m^2] = 0,$$

$$\Delta B_5^z[m^2] = -\frac{1}{120}\frac{-z}{12}\left[(\partial_\mu\partial_\nu m^2)(\partial^\mu\partial^\nu m^2) + (\partial^\mu m^2)(\partial_\mu\partial^2 m^2)\right].$$

$B_6^z[m^2]$: See Appendix (~1 page formula)

$a > 6$: DIY if needed

$$\left| d_\nu \ln \frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}} - s_\nu \right| < C\nu^{D-3-2a_{\max}}$$

$a_{\max} = 6$ → $D < 14$

$(D = 4)$ →

$$\left| d_\nu \ln \frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}} - s_\nu \right| < C\nu^{-11}$$

Analytic and numerical results

Fubini-Lipatov instanton

D-dimensional functional determinants

Functional determinant

$$\ln \frac{\det\left[-\partial^2 + \kappa\left(\frac{b}{|x|^2+b^2}\right)^2\right]}{\det\left[-\partial^2\right]} = \sum_{\nu} d_\nu \left[\ln \frac{\Gamma(\nu+1)\Gamma(\nu)}{\Gamma(\nu+1+z(\kappa))\Gamma(\nu-z(\kappa))}\right]$$

$b > 0$

$\kappa \geq -D(D+2)$

$z(\kappa) = -\frac{1}{2}(1-\sqrt{1-\kappa})$

Fubini-Lipatov instanton

D-dimensional functional determinants

Functional determinant

$$\ln \frac{\det\left[-\partial^2 + \kappa\left(\frac{b}{|x|^2+b^2}\right)^2\right]}{\det\left[-\partial^2\right]} = \sum_\nu d_\nu \left[\ln \frac{\Gamma(\nu+1)\Gamma(\nu)}{\Gamma(\nu+1+z(\kappa))\Gamma(\nu-z(\kappa))}\right]$$

$b > 0$

$\kappa \geq -D(D+2)$

$z(\kappa) = -\frac{1}{2}(1-\sqrt{1-\kappa})$

Large nu

$$\frac{\kappa}{4\nu} - \frac{\kappa^2}{96\nu^3} + \frac{\kappa^2(\kappa+2)}{960\nu^5} - \frac{\kappa^2(3\kappa^2+16\kappa+32)}{21504\nu^7} + \frac{\kappa^2(\kappa+4)(\kappa^2+6\kappa+24)}{46080\nu^9}$$
$$-\frac{\kappa^2(\kappa^4+16\kappa^3+136\kappa^2+640\kappa+1280)}{270336\nu^{11}} + \mathcal{O}(\nu^{-13})$$

Fubini-Lipatov instanton

D-dimensional functional determinants

Functional determinant

$$\ln \frac{\det\left[-\partial^2+\kappa\left(\frac{b}{|x|^2+b^2}\right)^2\right]}{\det\left[-\partial^2\right]} = \sum_\nu d_\nu \left[\ln \frac{\Gamma(\nu+1)\Gamma(\nu)}{\Gamma(\nu+1+z(\kappa))\Gamma(\nu-z(\kappa))}\right]$$

$b > 0$

$\kappa \geq -D(D+2)$

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Large nu

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$$-\frac{\kappa^2(\kappa^4+16\kappa^3+136\kappa^2+640\kappa+1280)}{270336\nu^{11}} + \mathcal{O}(\nu^{-13})$$

Subtraction $(a_{\max}=6,\ z=0)$

Large nu

$$s_\nu = d_\nu\left[-\sum_{a=1}^{6}\int_0^\infty dr\, r\mathcal{I}_{a,\nu}(0,r)B_a^{z=0}[m^2](r)\right]$$

Fubini-Lipatov instanton

D-dimensional functional determinants

Functional determinant

$$\ln \frac{\det\left[-\partial^2+\kappa\left(\frac{b}{|x|^2+b^2}\right)^2\right]}{\det\left[-\partial^2\right]} = \sum_\nu d_\nu \left[\ln \frac{\Gamma(\nu+1)\Gamma(\nu)}{\Gamma(\nu+1+z(\kappa))\Gamma(\nu-z(\kappa))}\right]$$

$b > 0$

$\kappa \geq -D(D+2)$

$z(\kappa) = -\frac{1}{2}(1-\sqrt{1-\kappa})$

Large nu

$$\frac{\kappa}{4\nu} - \frac{\kappa^2}{96\nu^3} + \frac{\kappa^2(\kappa+2)}{960\nu^5} - \frac{\kappa^2(3\kappa^2+16\kappa+32)}{21504\nu^7} + \frac{\kappa^2(\kappa+4)(\kappa^2+6\kappa+24)}{46080\nu^9} - \frac{\kappa^2(\kappa^4+16\kappa^3+136\kappa^2+640\kappa+1280)}{270336\nu^{11}} + \mathcal{O}(\nu^{-13})$$

Large nu

Subtraction $(a_{\max}=6,\ z=0)$

$$s_\nu = d_\nu\left[-\sum_{a=1}^{6}\int_0^\infty dr\, r\,\mathcal{I}_{a,\nu}(0,r)B_a^{z=0}[m^2](r)\right]$$

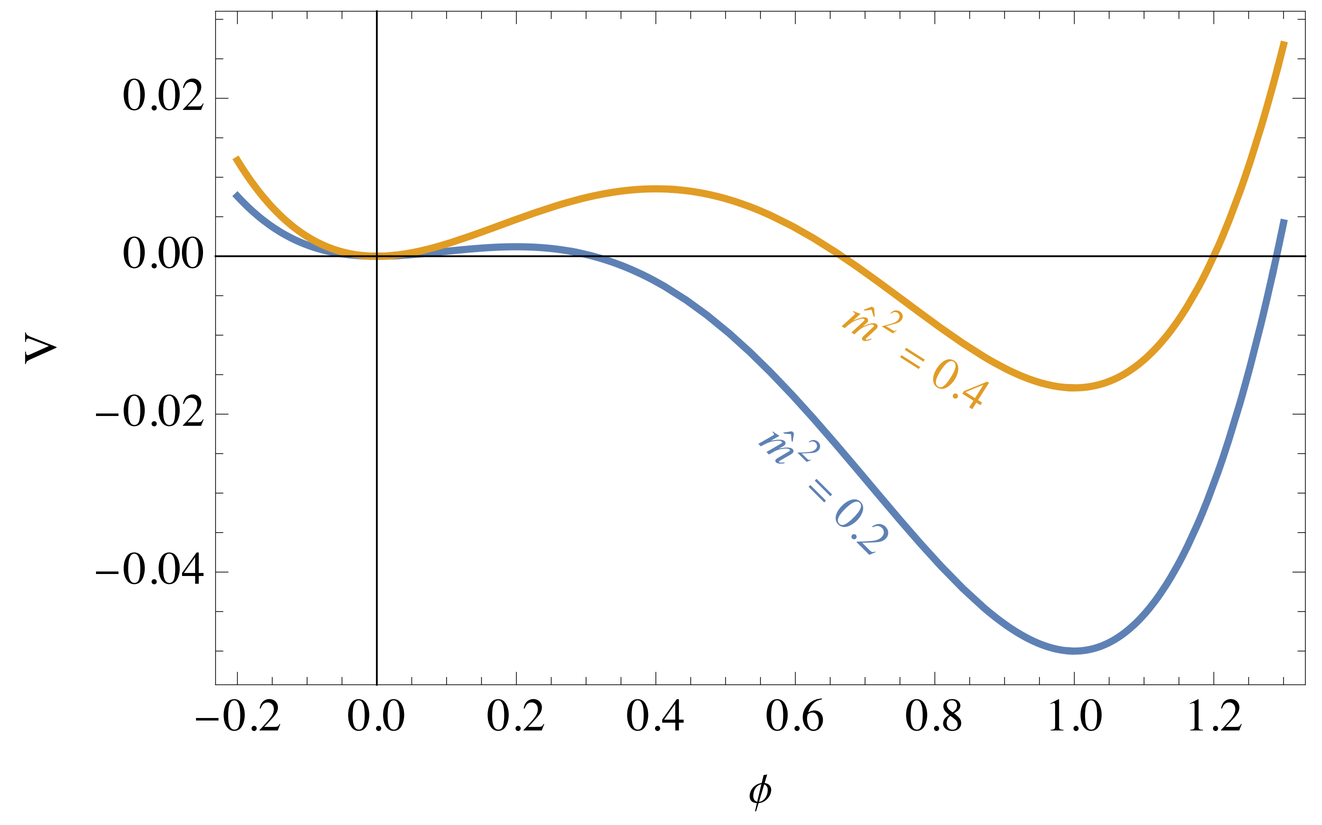
$d_\nu \sim \nu^{D-2}$ → Regularize $D < 14$

The regularized values of functional determinants are available in the paper

Numerical bounce

Four dimensions

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{\hat{m}^2 + v^2}{3v}\phi^3 + \frac{\hat{m}^2}{2}\phi^2 \qquad (v = 1)$$

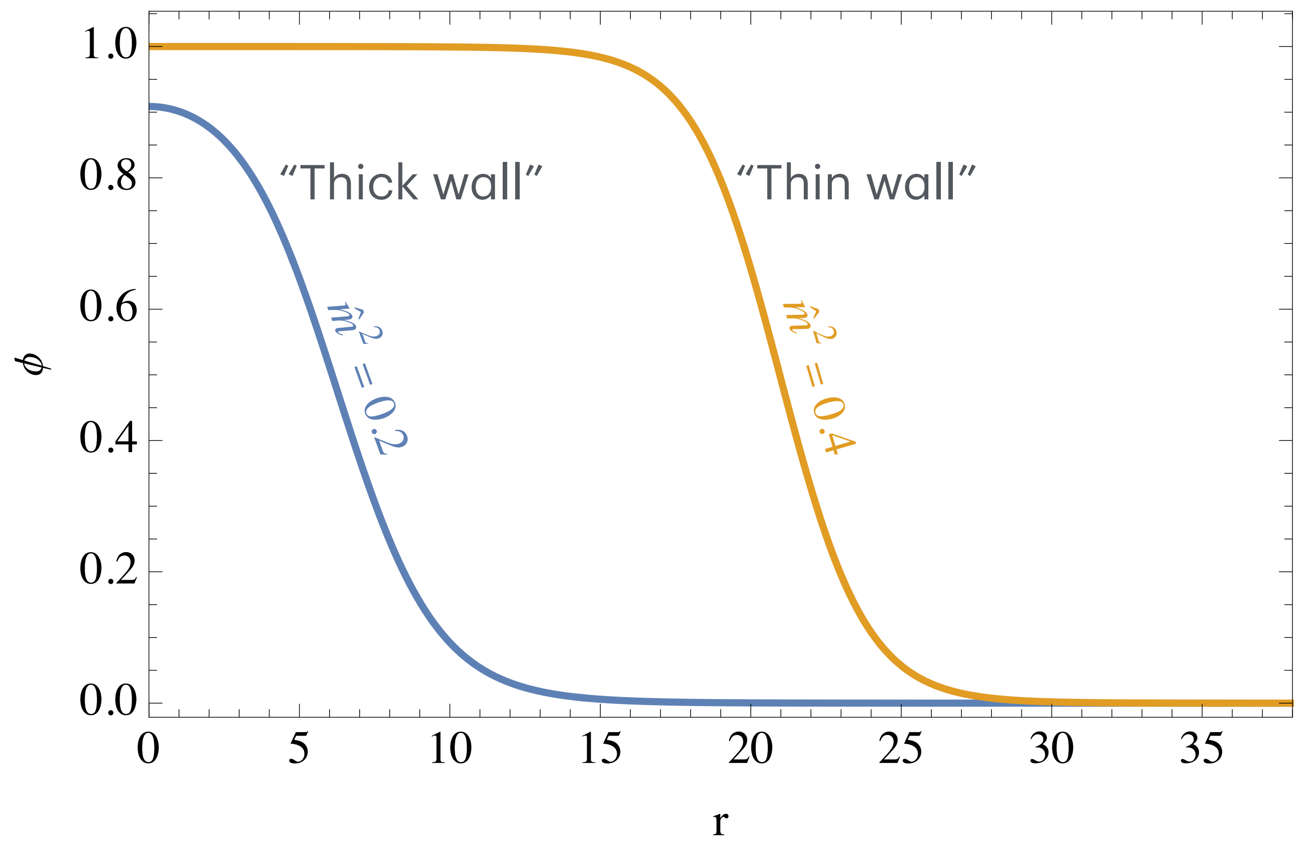


Bounce solution

$$\partial_r^2\phi + \frac{3}{r}\partial_r\phi = V'(\phi) \qquad \phi'(0) = 0,\ \phi(\infty) = 0$$

$$\ln\frac{\det[-\partial^2 + V''(\phi)]}{\det[-\partial^2 + V''(0)]} = \sum_\nu \left[d_\nu \ln\frac{\prod_i \lambda_{\nu i}}{\prod_i \hat{\lambda}_{\nu i}} - s_\nu \right] + S$$

$$= d_\nu \ln R_\nu|_{\text{fin}}$$



"Larger n, Smaller C"

$$\left| d_\nu \ln R_\nu|_{\text{fin}} \right| < C\nu^{-n}$$

Baacke-Lavrelashvili '04 (Feynman diagram)

Dunne-Kirsten '06 (WKB)

vs.

Shoji-Yamaguchi '25 (Heat kernel expansion)

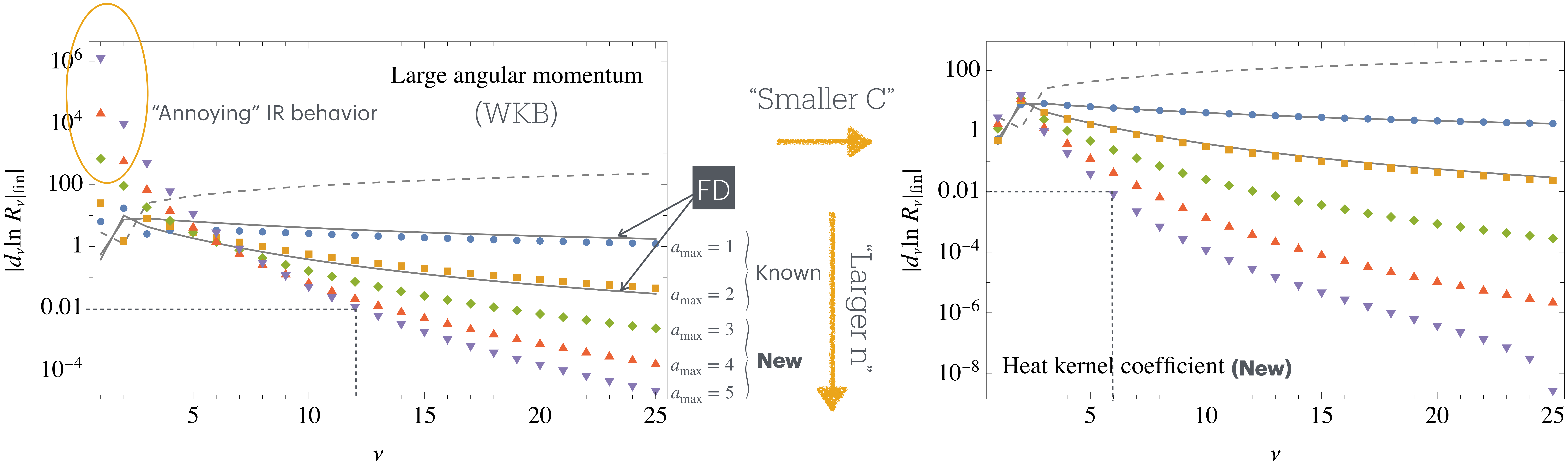
Numerical comparison

Thick wall

$\hat{m}^2 = 0.2$

$z = 0.2$ (Not optimal)

$$\left| d_\nu \ln R_\nu |_{\text{fin}} \right| < C\nu^{-n} \quad n = 3 - D + 2a_{\max}$$



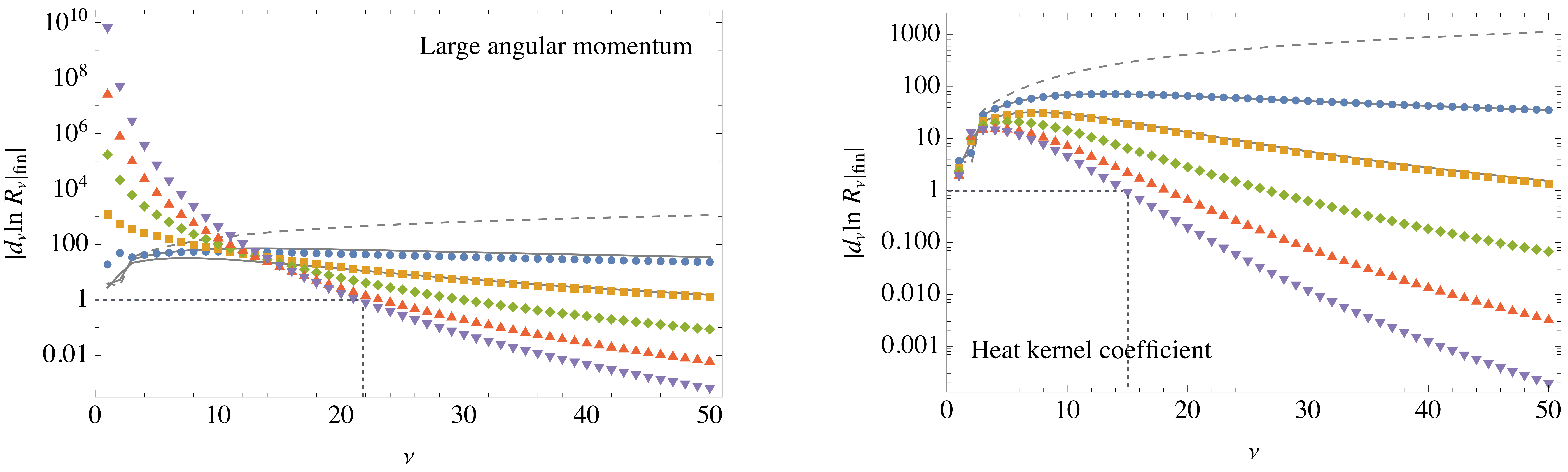
Numerical comparison

Thin wall

$\hat{m}^2 = 0.4$

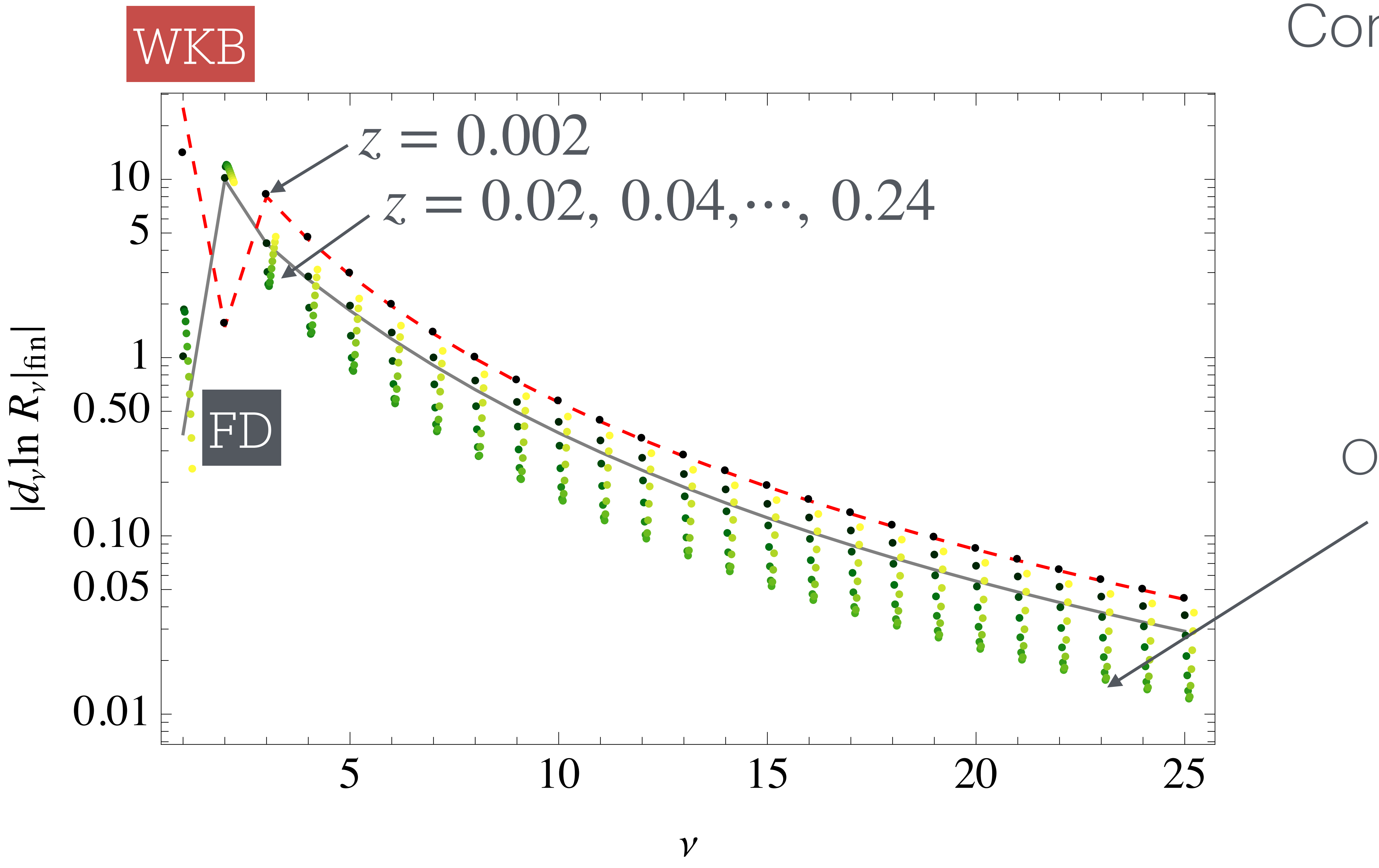
$z = 0.4$ (Not optimal)

$$\left| d_\nu \ln R_\nu |_{\text{fin}} \right| < C\nu^{-n} \quad n = 3 - D + 2a_{\max}$$



z-dependence

$\hat{m}^2 = 0.2 \qquad a_{\max} = 2$



Condition to minimize the next order

$$\sum_{a=1}^{a_{\max}} \frac{(-z)^{a_{\max}+1-a}}{(a_{\max}+1-a)!} \int_0^\infty \mathrm{d}r r^{2a_{\max}+1} \left(B_a[m^2 - z] - B_a[\hat{m}^2 - z] \right)$$

(Sub-leading part of the formula)

$$\sim \int_0^\infty \mathrm{d}r r^{2a_{\max}+1} \left(B_{a_{\max}+1}[m^2] - B_{a_{\max}+1}[\hat{m}^2] \right)$$

(Next order)

Optimal value

$z \simeq 0.13$

$a_{\max} = 2$ We can **minimize** the next order

$a_{\max} = 3$ We can **eliminate** the next order

Promoted to $a_{\max} = 4$ equivalent

Summary

	Derivation	Formulas	IR behavior (Small nu)	Convergence (Large nu)	Subtraction part	Adding-back part	Dimensions	Regularization Scheme
Baacke-Lavrelashvili (FD)	Easy ✓	Moderate	Good ✓	Good ✓	ODEs for each nu	Feynman diagram	D<6 Too hard to compute FDs	Dimensional ✓
Dunne-Kirsten (WKB)	Hard	Simple ✓	Bad	Moderate	Single integral ✓✓	Single integral ✓	D<6 Too hard to obtain the next order formula	Zeta function
Hur-Min (Radial WKB)	Very hard	Complicated	Good ✓	Good+ (Higher orders) ✓✓	Single integral for each nu (can be summed) ✓	Single integral ✓	General Explicit formulas for D<6 ✓	Zeta function (*)
★ **Shoji-Yamaguchi (HKC, z=0) "Improved WKB"**	Hard	Simple ✓	Moderate	Good+ (Higher orders) ✓✓	Single integral ✓✓	Zero ✓✓	General Explicit formulas for D<14 ✓✓	Both ✓✓
★ **Shoji-Yamaguchi (HKC, optimal z) "Improved FD"**	Hard	Simple ✓	Good ✓	Best+ (Higher orders) ✓✓✓	Single integral for each nu ✓	Single integral ✓	General Explicit formulas for D<14 ✓✓	Both ✓✓

Large nu

Straightforward generalization of HKC to multi-field, gauge/fermion ($D < 5$, $a_{\rm max} = 2$)