Thermodynamics from the S-matrix

Based on collaboration with Emanuele Gendy & Joan Elias Miro
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Introduction

1. Thermodynamics is interesting for high-energy physicists mainly because of cosmology, where $T \sim T_{h.e.}$

2. Ideally explore phases of matter at very high temperatures, understand phase transitions (non-trivial dynamics and rich of consequences, e.g. gravitational waves)

3. Often enough to treat cosmic plasma as in the free theory limit (counting of d.o.f.)

4. Sometimes interactions start to matter, e.g. axion production rate at $T \gtrsim T_{\mathrm{QCD}}$ is sensitive to the behaviour of $f(g_s)$ of the quark-gluon plasma

5. Need to compute higher-order effects

Introduction

1. The standard method consists in calculating Feynman diagrams in Euclidean space with compact time dimension $[0,\beta]$ and (anti-)periodic b.c.

2. A physically more transparent method is available which is Lorentzian and interprets the free energy in terms of scattering among particles in the thermal bath.

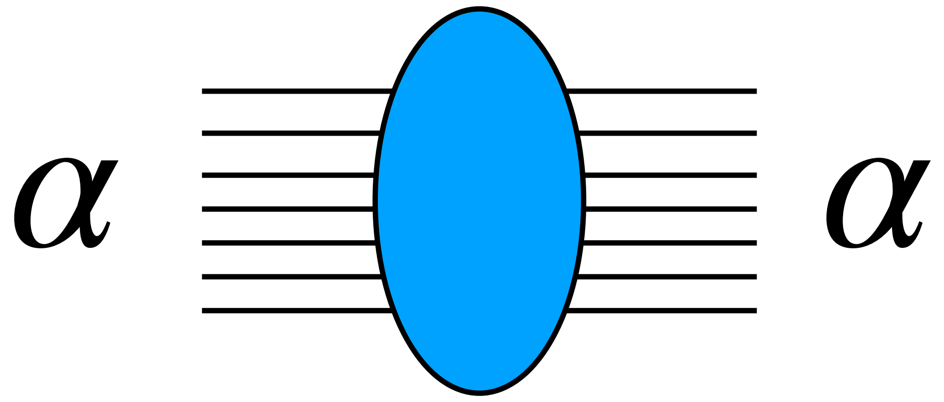
3. From the free energy one can in principle obtain other quantities (add a current $\int j(x)O(x)$ and take j derivatives)

4. Purely S-matrix method could bypass a Lagrangian formulation: (i) synergy with amplitude methods; (ii) use input from experiments

DMB method

Dashen, Ma, Bernstein

- Compact formula for the partition function



$$Z = Z_0 + \frac{\beta}{2\pi i} \sum_{\alpha} \int \mathrm{d}E\, e^{-\beta E} \langle \alpha \,|\, \ln S(E) \,|\, \alpha \rangle$$

a. Derive general results from formal manipulations

b. Envisage an expansion and work order by order. Is it consistent? Powerful?

Free theory free energy

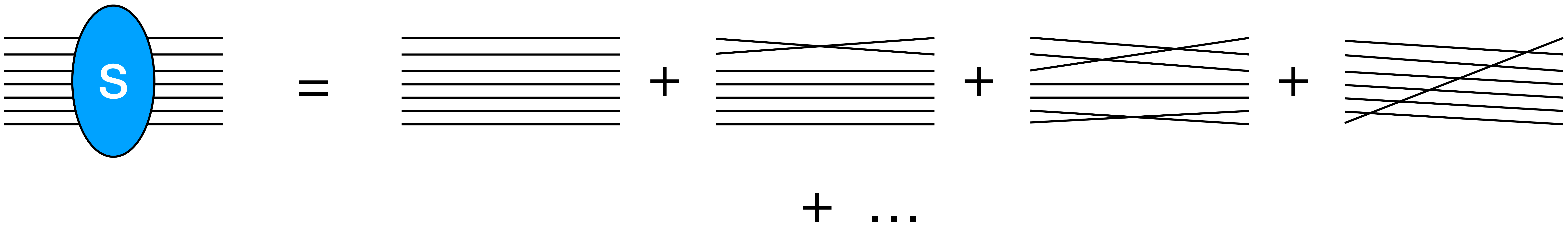
S-matrix interpretation

$$Z = e^{-\beta F} = \sum_{\text{states}} \frac{\langle \text{state} \,|\, e^{-\beta H_0} \,|\, \text{state} \rangle}{\langle \text{state} \,|\, \text{state} \rangle}$$

Free theory free energy

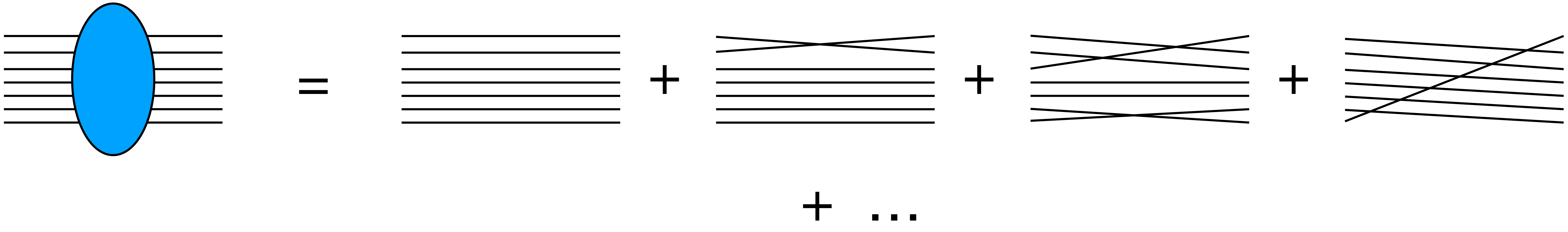
- In particle physics, states are n-particle Fock states

$$Z = \underbrace{\sum_{\alpha}}_{\sum_n \int \mathrm{d}\Phi_n} e^{-\beta E_\alpha} \underbrace{\langle \alpha | \alpha \rangle}_{S_{\alpha\alpha}}$$



Free theory free energy

- Free theory is not trivial because of possible exchange effects



- Free energy = log Z comes from “histories” where all particles get exchanged
- Fermions pick a minus for each crossing and one gets the standard

$$F_0(\beta) = \beta^{-1} V_{(d)} \int \mathrm{d}^d k \ln\Big(1 - e^{-\beta E_{\mathbf{k}}}\Big)/(2\pi)^d$$

+ for fermions

Free theory lessons

- $F \sim \ln Z$ from "connected histories", analogous to $W[J(x)] = \ln Z[J(x)]$

- Amplitudes can be disconnected in the S-matrix sense (cluster decomposition or diagrammatically)

- Distinguishable particles cannot get trace connected in the free theory

$$Z = \Sigma_\alpha e^{-\beta E_\alpha} S_{\alpha\alpha}$$

- Gives the partition function for the free theory
- The idea is now to include histories where particles interact by making S the full S-matrix
- Many reasons why this formula is too naive, e.g. not a real object. How to correctly account for histories with scattering was taught us by DMB

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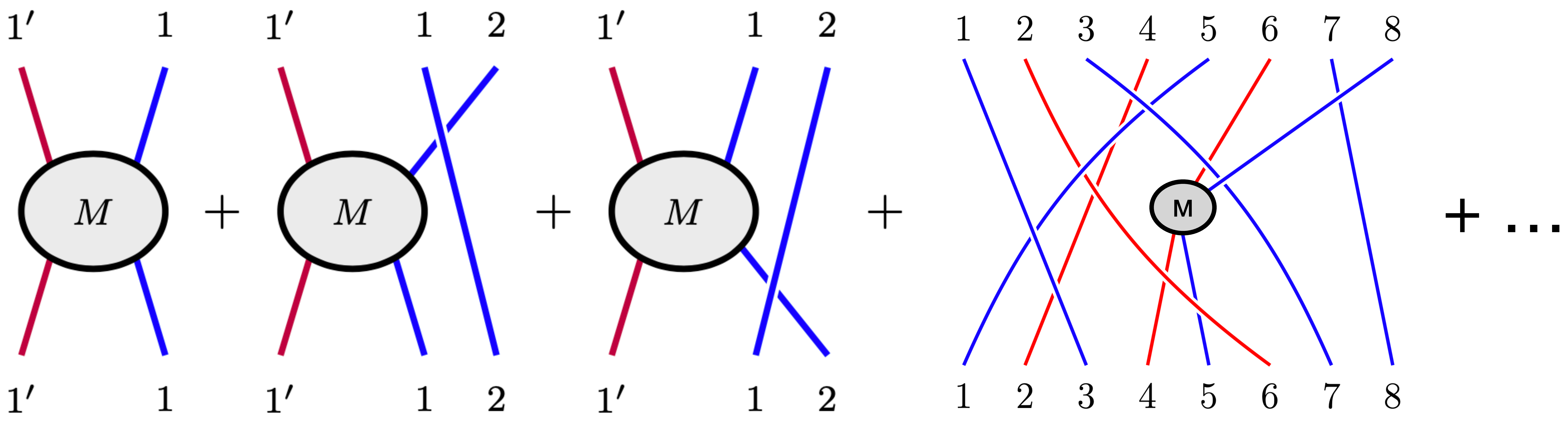
Leading effect of interactions

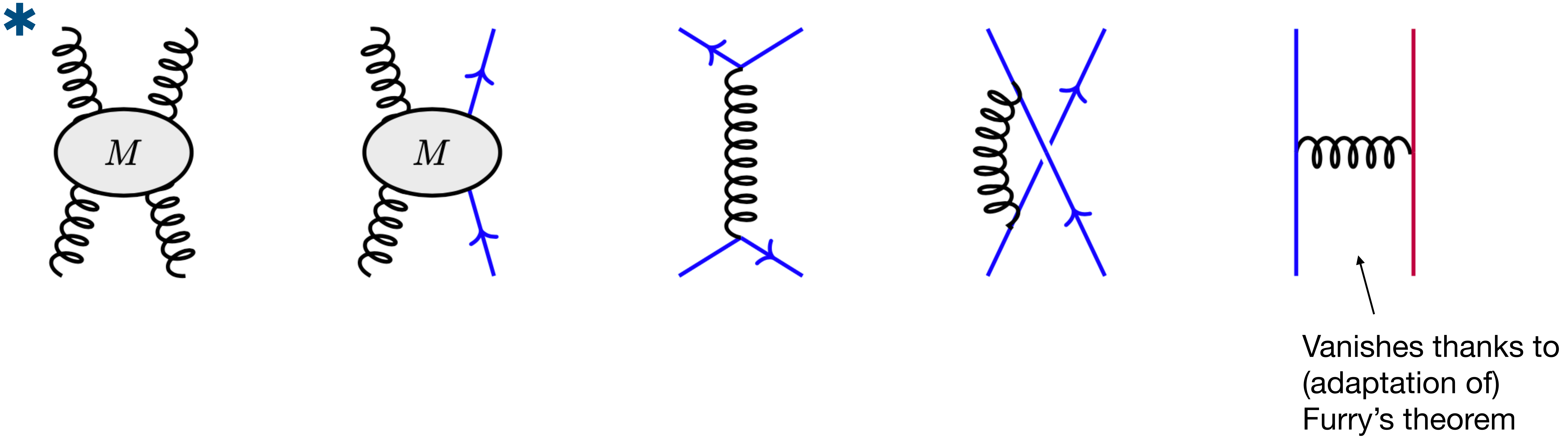


- 2 to 2 scattering at tree level + arbitrary number of free propagating

- Can connect different species via $M_{a,b\to a,b}$

connected by trace

$$\Delta f = -\frac{1}{2}\sum_{I,J}\int \frac{\mathrm{d}^d k_I}{2E_I(2\pi)^d}\int \frac{\mathrm{d}^d k_J}{2E_J(2\pi)^d}\, n(E_I)\, n(E_J)\, M_{IJ\to IJ}(\mathbf{k}_I,\mathbf{k}_J)$$



- Contributions with dangerous t-channel gluons ($t \to 0$ in the forward limit) cancel upon summing over colours

- Well known amplitudes with simple forward limit

$$* \quad M_{\text{Parke–Taylor}} = \frac{\langle 12\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \to 1$$

no net helicity +
dimensional analysis

$$\boxed{Z = Z_0 + \frac{\beta}{2\pi i} \sum_{\alpha} \int \mathrm{d}E\, e^{-\beta E} \langle \alpha \,|\, \ln S(E) \,|\, \alpha \rangle}$$

- At higher orders (starting from $3 \to 3$ amplitudes) it becomes crucial to understand the meaning of S(E). First, qualitatively from the mathematical point of view:

(well-defined) distribution in α, β

$$\lim_{E \to E_\alpha} \underbrace{\langle \beta \,|\, S(E) \,|\, \alpha \rangle}_{\text{distribution in } \alpha, \beta, E} = S_{\beta\alpha}$$

$$\lim_{E \to E_\alpha} \langle \alpha | S(E) | \alpha \rangle = S_{\alpha\alpha}$$

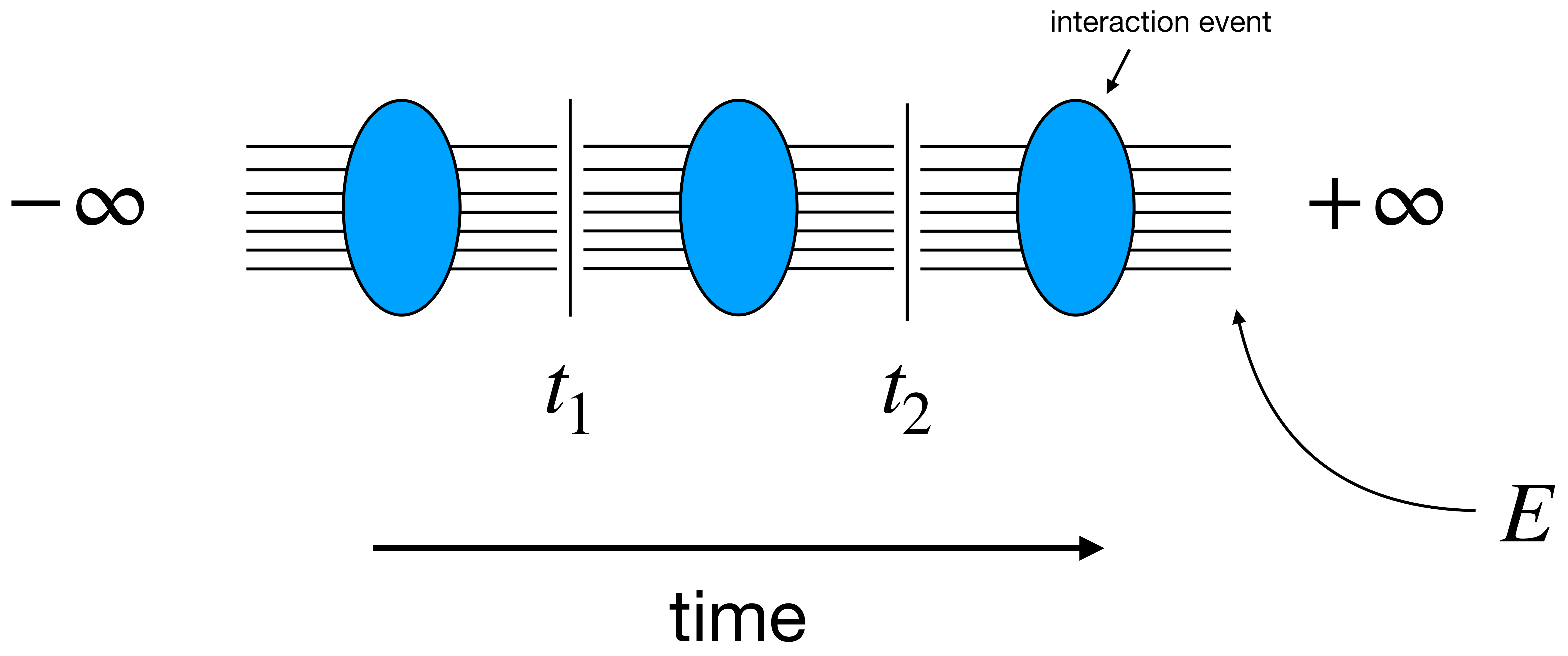
- In the forward configuration the limit is not always well defined
- The full E-dependence has to be kept, at least in a neighbourhood of E_α
- $S(E) = 1 + 2\pi i \delta(E - H_0) T(E)$

$$T(E) \equiv V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \ldots$$

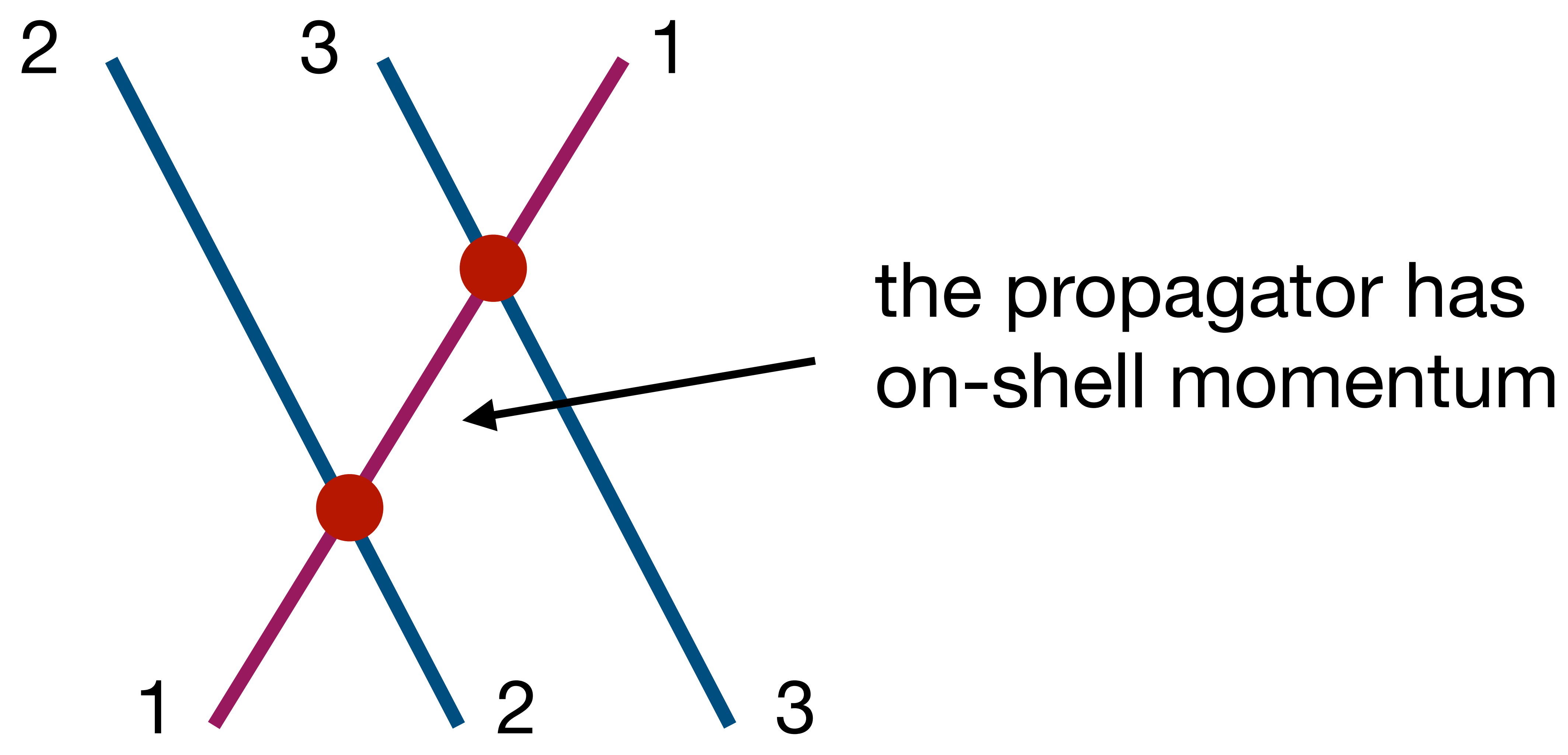
Lippmann-Schwinger

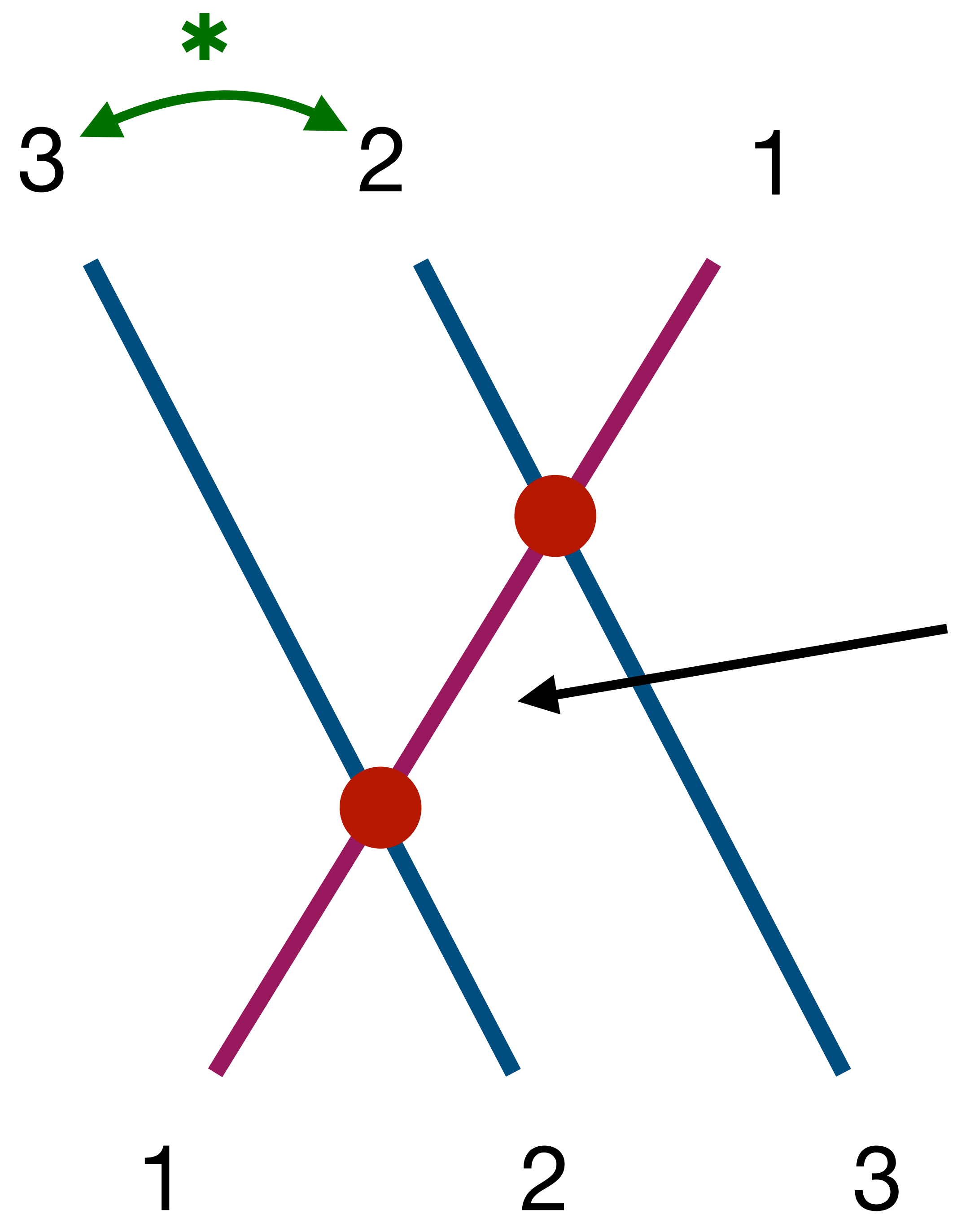
$V = H - H_0$

- Physically, the problem comes from contributions where the intermediate state is identical to the asymptotic one, α



Next orders ($\lambda\phi^4$)



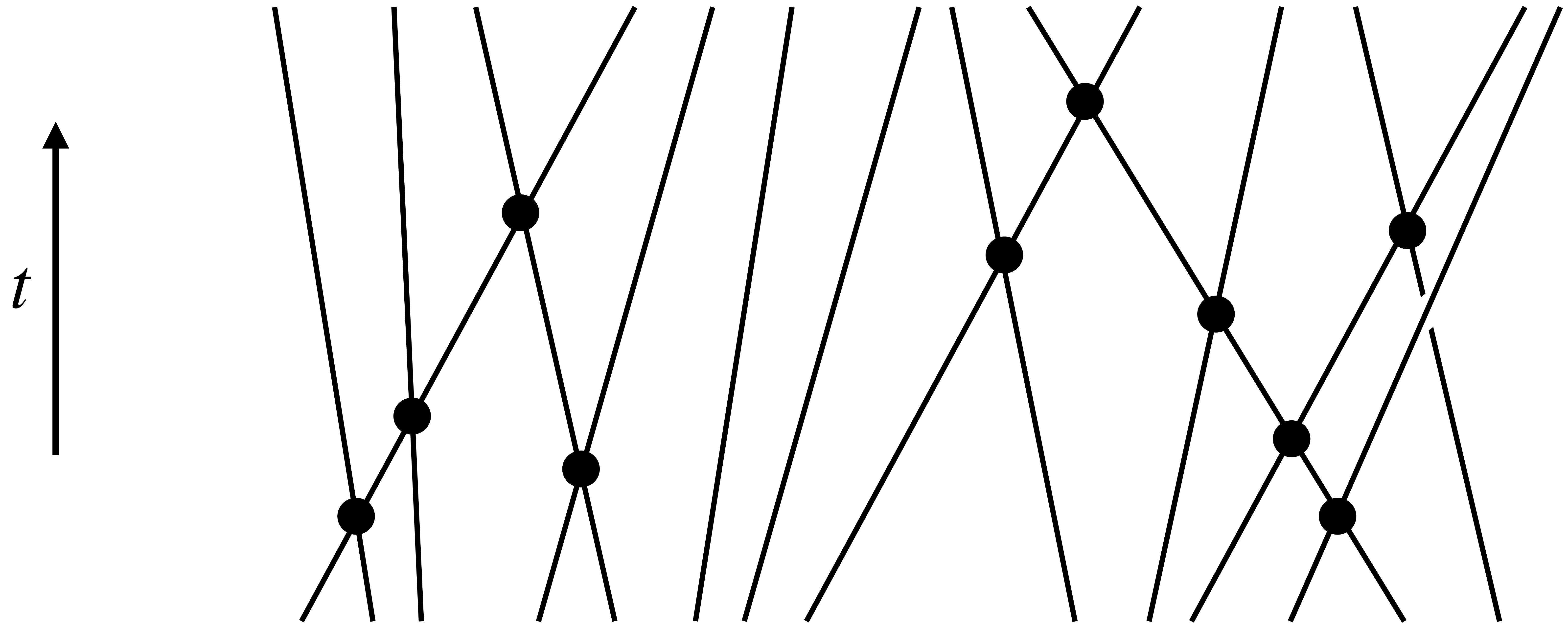


with this * trace identification
the propagator is non-singular

$$\frac{1}{(p_1+p_2-p_3)^2+i\epsilon}$$

$$F = -V\frac{1}{3!}\int \mathrm{d}\Phi_{\mathrm{th}}(3)\,\mathscr{R}[M_{3\to 3}^{\text{forward}}]$$

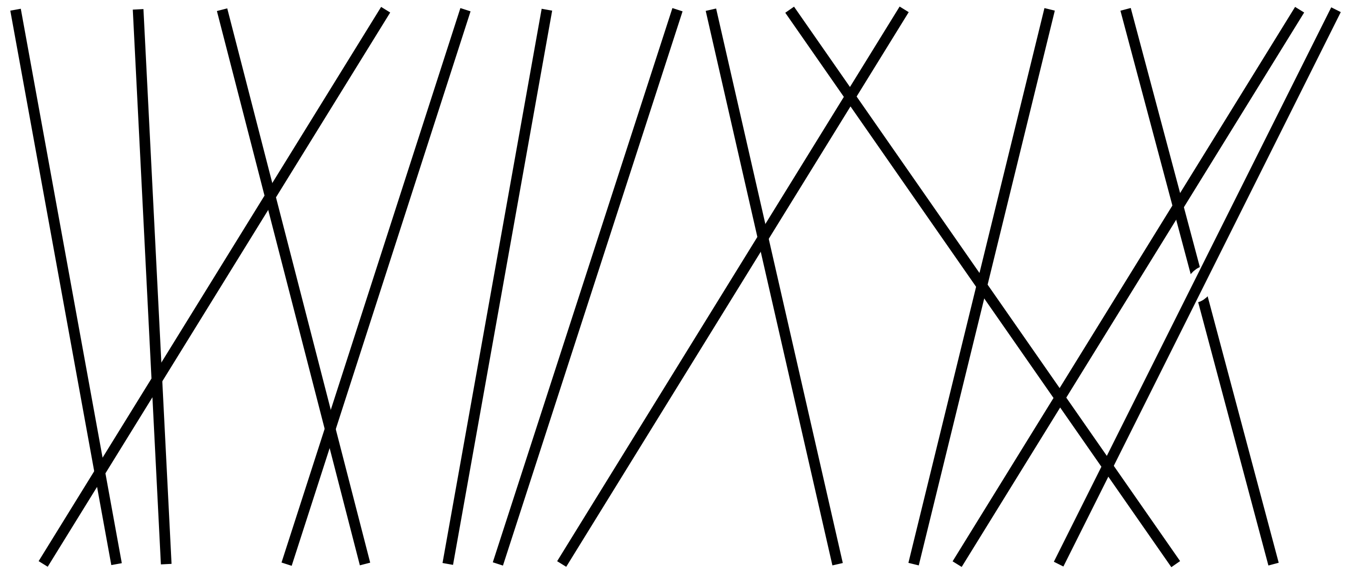
Most singular contributions



IR resummation

- Despite being individually IR divergent, their sum is IR safe

$$\sum_{\text{these histories}}$$



$$= \begin{cases} f = \dfrac{1}{\lambda}\displaystyle\int_0^{\lambda} \mathrm{d}\ell\, g(\ell) \\ g(\lambda) = \displaystyle\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1 - e^{-\sqrt{k^2+m^2}}\right) \\ m^2 \equiv \lambda \displaystyle\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{n_B\left(\sqrt{p^2+m^2}\right)}{2\sqrt{p^2+m^2}} \end{cases}$$

recursive expression

Dolan & Jackiw 1973

- Formulas can be tested against known results in the high T limit

$$m^2 = \lambda\left(\frac{1}{24} - \frac{m}{8\pi} + \dots\right) = \frac{\lambda}{24} - \frac{\lambda^{3/2}}{8\pi\sqrt{24}} + O(\lambda^2)$$

$$g(\lambda) = -\frac{\pi^2}{90} + \frac{m^2}{24} - \frac{m^3}{12\pi} + \dots = -\frac{\pi^2}{90} + 2\,\frac{\lambda}{1152} - \left(\frac{5}{2}\right)\frac{\lambda^{3/2}}{576\sqrt{6}\,\pi} + O(\lambda^2)$$

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- Same phenomenon in QCD leads to bad convergence of $f(g_s)$…

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